1. Is \( p \rightarrow (q \lor r) \equiv (p \rightarrow q) \lor (p \rightarrow r) \)? Complete the truth table below and explain your conclusion.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>p \rightarrow q</th>
<th>p \rightarrow r</th>
<th>q \lor r</th>
<th>(p \rightarrow (q \lor r))</th>
<th>(p \rightarrow q) \lor (p \rightarrow r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

They are logically equivalent since they have the same truth values for all possible values of \( p, q, \) and \( r \).

2. Consider the argument form:

\[
\begin{align*}
p & \\
p \rightarrow q & \\
\sim q \lor r & \\
\therefore r &
\end{align*}
\]

Use the truth table below to determine whether this is a valid argument form. Clearly explain how you made your decision.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>\sim q</th>
<th>p \rightarrow q</th>
<th>\sim q \lor r</th>
<th>CRITICAL ROW</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>TRUE</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>TRUE</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>TRUE</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>FALSE</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>TRUE</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>TRUE</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>TRUE</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>FALSE</td>
</tr>
</tbody>
</table>

The argument form is valid if the conclusion \( r \) is true whenever all the hypotheses are true. All hypotheses are only satisfied when \( p, q, \) and \( r \) are all true. Hence \( r \) is true and the argument is valid.

3. Determine whether the following argument is valid or invalid. If it is valid, state the rule of inference (modus ponens or modus tollens); if it is invalid, state whether converse or inverse error is made.

If this number is larger than two, then its square is larger than four.
This number is not larger than two.
Therefore, the square of this number is not larger than four.

**Argument Form is:**

\[
\begin{align*}
p & \\
p \rightarrow q & \\
\sim p & \\
\therefore \sim q &
\end{align*}
\]

**P → q** is NOT valid — **Inverse Error**

4. Let \( m, n \in \mathbb{Z} \). Let \( R(m, n) \) be the predicate “If \( m \) is a factor of \( n^2 \) then \( m \) is a factor of \( n \).

(a) Is \( R(25, 10) \) true or false? Explain.
(b) Find values for \( m \) and \( n \) for which \( R(m, n) \) is true.

\[R(25, 10)\] is: IF 25 is a factor of \( 10^2 \), THEN 25 is a factor of 10.

FALSE. "25 is a factor of 100 (true)" but "25 is a factor of 10" (false), so implication is \( T \rightarrow F \) which is FALSE.

\[R(4, 8)\] is TRUE.

February 15, 2007
5. For the statement “Everyone at the park was beautiful or smart”
   (a) Translate the statement into formal logic using the appropriate logical symbols and predicates.
   (b) Give the formal negation of the statement and simplify.
   (c) Give an informal translation of the formal negation.

6. Let \( P \) be the set of all people. Let \( p, q \in P \). Let \( F(p, q) \) be \( "p \) is the (biological) father of \( q"\).
   (a) Give an informal translation of: \( \forall q, \exists p, F(p, q) \).
   (b) Give an informal translation of: \( \exists q, \forall p, F(p, q) \).
   (c) Which if any of the previous statements are true? Give a clear explanation.

7. Complete the argument using universal modus ponens or universal modus tollens as indicated.
   (a) (modus tollens)
   All healthy people eat an apple a day.
   Mark ____________________________________________.
   \therefore _______________________________________.
   (b) (modus ponens)
   All healthy people eat an apple a day.
   Mark ____________________________________________.
   \therefore _______________________________________.

8. Write \( 83_{10} \) in both eight-bit binary and hexadecimal form.

9. Write \( -83_{10} \) in eight-bit binary form using twos-complement representation.
5a. (LET: B(x) BE "x IS BEAUTIFUL"
     S(x) BE "x IS SMART"

     LET THE DOMAIN BE ALL PEOPLE AT THE PARK.

     SO STATEMENT BECOMES:

     \[ \forall x (B(x) \lor S(x)) \]

5b. \(~(\forall x (B(x) \lor S(x))) \equiv \exists x (~B(x) \land S(x))\)

5c. "SOMEONE AT THE PARK WAS NEITHER BEAUTIFUL NOR SMART."

6a. "FOR EACH PERSON Q, THERE IS A PERSON P SO THAT P IS THE FATHER OF Q"
     OR
     "EVERYBODY HAS A (BIOLOGICAL) FATHER"

6b. "THERE IS A PERSON Q, SO THAT FOR ALL PEOPLE P, P IS THE FATHER OF Q"
     OR
     "THERE IS SOMEONE FOR WHOM EVERYONE IS THEIR FATHER."

6c. STATEMENT "A" IS TRUE, STATEMENT "B" IS FALSE DUE TO PROCESS OF
     SEXUAL REPRODUCTION.

7a. (MODUS TOLENS)
     ALL HEALTHY PEOPLE EAT AN APPLE A DAY.
     MARK DOES NOT EAT AN APPLE A DAY.
     \[ \therefore \text{MARK IS NOT A HEALTHY PERSON} \]

7b. (MODUS PONENS)
     ALL HEALTHY PEOPLE EAT AN APPLE A DAY.
     MARK IS A HEALTHY PERSON.
     \[ \therefore \text{MARK EATS AN APPLE A DAY}. \]

8. \(32_{10} = 64_{10} + 16_{10} + 2_{10} + 1_{10} = 01010011_2 = 53_{16}\)

9. \(-83_{10} = \boxed{10101101}_2\) (FLIP BITS IN BINARY)
     (REP OF 83 AND ADD 1)
1. Determine whether the statement below is true or false. Justify your answer with a proof or counterexample, as appropriate.

\[ \forall n \in \mathbb{Z}, n^2 - n + 11 \text{ is a prime number} \]

2. Give a formal, well-formed direct proof of the proposition "The difference of any even integer minus any odd integer is an odd integer." Be clear and complete.

3. Given the statement "The product of any non-zero rational number and any irrational number is an irrational number"

   (a) Write the negation of the statement.
   (b) Prove the proposition by contradiction.

4. Given the proposition: \( \forall n \in \mathbb{Z}, 3n + 2 \) odd \( \rightarrow n \) odd

   (a) Write the contrapositive statement.
   (b) Prove the proposition by contraposition.

5. Prove that \( \sqrt{5} \) is irrational. Hint: try contradiction.

6. Give an example to show that if \( d \) is not prime and \( n^2 \) is divisible by \( d \), then \( n \) need not be divisible by \( d \).

---

1. **False. Consider \( n = 11 \).**

   \[ 11 \in \mathbb{Z} \text{ but } 11^2 - 11 + 11 = 11^2 \text{ is not prime since } 11^2 = 11 \times 11, \text{ hence a product of two integers neither of which is one.} \]

2. **Let \( n \) be a particular but arbitrarily chosen even integer; let \( m \) be a particular but arbitrarily chosen odd integer.**

   \( n \) even \( \rightarrow \exists p \in \mathbb{Z} \text{ s.t. } n = 2p \)

   \( m \) odd \( \rightarrow \exists q \in \mathbb{Z} \text{ s.t. } m = 2q + 1 \)

   Hence \( n - m = 2p - (2q + 1) = 2p - 2q - 1 = 2(p - q - 1) + 1 \)

   Now, \( p, q, 1 \in \mathbb{Z} \), and \( \mathbb{Z} \) closed under subtraction \( \rightarrow p - q - 1 = r \in \mathbb{Z} \)

   \( r = (p - q - 1) \in \mathbb{Z} \)

   \( n - m = 2r + 1, \quad r \in \mathbb{Z} \)

   \( \rightarrow n - m \) is an odd integer (by defn of odd integer).

   QED.
3a. "THE PRODUCT OF ANY NON-ZERO RATIONAL NUMBER AND ANY IRRATIONAL NUMBER IS AN IRRATIONAL NUMBER."

\[ a \in \mathbb{Q}, \forall b \in \mathbb{R} - \mathbb{Q}, \quad ab \in \mathbb{R} - \mathbb{Q} \]

The negation is: \( \exists a \in \mathbb{Q}^*, \forall b \in \mathbb{R} - \mathbb{Q} \) s.t. \( ab \in \mathbb{Q} \)

b. Pf by contradiction. Suppose not. Then \( \exists a \in \mathbb{Q}^*, \forall b \in \mathbb{R} - \mathbb{Q} \) s.t. \( ab \in \mathbb{Q} \)

\[ a \in \mathbb{Q}^* \Rightarrow \exists p, q \in \mathbb{Z}, \; p \neq 0, q \neq 0 \; \text{s.t.} \; a = \frac{p}{q} \]
\[ ab \in \mathbb{Q} \Rightarrow \exists r, s \in \mathbb{Z}, \; s \neq 0 \; \text{s.t.} \; ab = \frac{r}{s} \]

(1) and (2) \( \Rightarrow \frac{p}{q} \cdot b = \frac{r}{s} \Rightarrow b = \frac{pr}{qs} \).

Now, \( r, q, p, s \in \mathbb{Z}, \; p \neq 0, s \neq 0 \Rightarrow m = qr \in \mathbb{Z}, m = ps \in \mathbb{Z}, \; ps \neq 0 \)

(by closure prop. of \( \mathbb{Z} \))

Hence \( b = \frac{n}{m}, \; n, m \in \mathbb{Z}, \; m \neq 0 \)

\( \Rightarrow b \in \mathbb{Q} \) but by hypothesis \( b \in \mathbb{R} - \mathbb{Q} \)

\( \therefore \) Supposition is false so Theorem is proved. QED.

4. \( \forall n \in \mathbb{Z}, \; 3n+2 \text{ odd} \Rightarrow n \text{ odd} \)

a. The contrapositive is \( \forall n \in \mathbb{Z}, \; n \text{ even} \Rightarrow 3n+2 \text{ even.} \)

b. Pf of statement in part a.

\( n \text{ even } \Rightarrow \exists k \in \mathbb{Z} \; \text{s.t.} \; n = 2k \).

\( \therefore \; 3n+2 = 3(2k) + 2(3k+1) \). Now, \( 3, k, 1 \in \mathbb{Z} \)

\( \Rightarrow m = 3k+1 \in \mathbb{Z} \) (by closure axioms of \( \mathbb{Z} \))

Hence \( 3n+2 = 2m, \; m \in \mathbb{Z} \)

\( \therefore \; 3n+2 \text{ is even} \) (by defn. of even)

QED
5. Prove $\sqrt{5}$ is irrational.

(Lemma: $\forall n \in \mathbb{Z}, \; 5n^2 \rightarrow 5n$)

**Pf by contradiction:** We prove the logically equivalent

$\forall n \in \mathbb{Z}, \; 5n \rightarrow 5n^2$.

Let $n \in \mathbb{Z}$ s.t. $5n$. This implies (by the Quotient-Remainder Theorem), $\exists q \in \mathbb{Z}, \exists r \in \mathbb{Z}, \; 1 \leq r \leq 4$ s.t. $n = 5q + r$.

**Case 1, $r = 1$**

$n = 5q + 1 \rightarrow n^2 = 25q^2 + 10q + 1 = 5(5q^2 + 2q) + 1$. Now

$m = 5q^2 + 2q \in \mathbb{Z}$ by closure axioms of $\mathbb{Z}$. Hence $n^2 = 5m + 1$ \hspace{1cm}$\Rightarrow$ \hspace{1cm} $5 \nmid n^2$.

**Case 2, $r = 2$**

$n = 5q + 2 \rightarrow n^2 = 25q^2 + 20q + 4 = 5(5q^2 + 4q) + 4$. Now

$m = 5q^2 + 4q \in \mathbb{Z}$ (closure), hence $n^2 = 5m + 4 \Rightarrow 5 \nmid n^2$.

**Case 3, $r = 3$**

$n = 5q + 3 \rightarrow n^2 = 25q^2 + 30q + 9 = 5(5q^2 + 6q + 1) + 4$. Now

$m = 5q^2 + 6q + 1 \in \mathbb{Z}$ (closure), hence $n^2 = 5m + 4 \Rightarrow 5 \nmid n^2$.

**Case 4, $r = 4$**

$n = 5q + 4 \rightarrow n^2 = 25q^2 + 40q + 16 = 5(5q^2 + 8q + 3) + 1$. Now

$m = 5q^2 + 8q + 3 \in \mathbb{Z}$ (closure), hence $n^2 = 5m + 1 \Rightarrow 5 \nmid n^2$.

$\forall n \in \mathbb{Z}, \; 5 \nmid n^2 \Rightarrow$ equivalently $\forall n \in \mathbb{Z}, \; 5 \nmid n^2$.

**Pf that $\sqrt{5}$ is irrational by contradiction.**

Suppose not. Then $\sqrt{5}$ is rational $\Rightarrow \exists m, n \in \mathbb{Z}, \; m \neq 0$ s.t.

$\sqrt{5} = \frac{m}{n}$ \hspace{1cm} $\Rightarrow$ \hspace{1cm} $5 = \frac{n^2}{m^2}$ \hspace{1cm} $\Rightarrow$ \hspace{1cm} $n^2 = 5m^2$ \hspace{1cm} $\Rightarrow$ \hspace{1cm} $n^2 \in \mathbb{Z}$ (closure) and $n^2 = 5m^2$; by definition of divisibility, $5 \mid n^2 \Rightarrow 5 \mid n$ (by Lemma)

Hence $\exists k \in \mathbb{Z}$ s.t. $n = 5k$. Subst. into $\frac{m}{n}$ gives $25k^2 = 5m^2$ \hspace{1cm} $\Rightarrow$ \hspace{1cm} $m^2 = 5k^2$ \hspace{1cm} $\Rightarrow$ \hspace{1cm} $5 \mid m^2$ (by Lemma) \hspace{1cm} $\Rightarrow$ \hspace{1cm} $5 \mid m$.

Therefore $\exists l \in \mathbb{Z}$ s.t. $m = 5l$

This contradicts $m$ and $n$ have no common factors. Hence $\sqrt{5}$ is irrational.
6. Non-prime \( d \) with \( d \nmid n^2 \) but \( d \mid n \).

For instance, consider \( d = 4, \ n = 6 \).

\( 4 \nmid 6 \) but \( 4 \nmid 6^2 \) (since \( 36 = 9 \cdot 4 \)).
Work the problems below in a neat, complete, organized manner.

1. Summation notation:
   (a) Express $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$ using summation (sigma) notation.
   (b) Expand the sum $\sum_{i=3}^{n} (i - 1)i$, including the first three and last terms.

2. Let $P(n)$ be the statement $\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$.
   (a) Is $P(2)$ true? Explain.
   (b) Write out in expanded form the statement $P(k)$, including the first two and last factors in the product on the left of the equals sign.
   (c) In proving that $P(n)$ is true for all $n \geq 2$ using mathematical induction, we must show the base case $P(2)$ is true and then show that $P(k) \rightarrow P(k+1), \; \forall k \geq 2$. Do this.

3. Consider the sequence $b_0, b_1, b_2, \ldots$ defined recursively as:
   
   $b_0 = 1$
   $b_k = 2b_{k-1} + 3, \; \forall k \geq 1$

   (a) Write down the next five terms of the sequence $(b_1, \ldots, b_5)$.
   (b) Looking at the terms $b_0, \ldots, b_5$ guess an explicit formula for the sequence. Hint: Consider the differences between successive terms.
   (c) Simplify the formula derived above using one of the identities below:

   $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$
   $1 + r + r^2 + \cdots + r^m = \frac{r^{m+1} - 1}{r - 1}$

4. A sequence $a_0, a_1, a_2, \ldots$ satisfies the recurrence relation

   $a_0 = 5, a_1 = 3$
   $a_k = 6a_{k-1} - 9a_{k-2}, \; \forall k \geq 2$

   (a) Find the next three terms $(a_2, a_3, a_4)$ of the sequence.
   (b) Find an explicit formula for $a_k$.
   (c) Verify that the formula for $k = 4$ and the value for $a_4$ from part (a) of this problem agree.

April 26, 2007
1a. \( \sum_{n=1}^{6} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \)

b. \( \sum_{i=3}^{n} (i-1)i = 2.3 + 3.4 + 4.5 + \ldots + (n-1)n \)

2. \( P(n) \Rightarrow \prod_{i=2}^{n} (1 - \frac{1}{i^2}) = \frac{n+1}{2n} \)

a) \( P(2) : \prod_{i=2}^{2} (1 - \frac{1}{i^2}) = \frac{3}{4} \) or \( 1 - \frac{1}{4} = \frac{3}{4} \) TRUE!

b) \( P(k) : \prod_{i=2}^{k} (1 - \frac{1}{i^2}) = \frac{k+1}{2k} \) or \( (1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \ldots (1 - \frac{1}{k^2}) = \frac{k+1}{2k} \)

c) \( P(k+1) \) is statement \( \prod_{i=2}^{k+1} (1 - \frac{1}{i^2}) = \frac{(k+1)+1}{2(k+1)} \) or
\( (1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \ldots (1 - \frac{1}{k^2})(1 - \frac{1}{(k+1)^2}) = \frac{k+2}{2(k+2)} \). WANT TO SHOW \( P(k) \Rightarrow P(k+1) \)

START WITH \( P(k) \) IN PART b). MULTIPLY BOTH SIDES BY \( (1 - \frac{1}{(k+1)^2}) \) TO GET:
\( (1 - \frac{1}{2^2}) \ldots (1 - \frac{1}{k^2})(1 - \frac{1}{(k+1)^2}) = \frac{k+1}{2k} \left( 1 - \frac{1}{(k+1)^2} \right) \)

\( \Rightarrow \prod_{i=2}^{k+1} (1 - \frac{1}{i^2}) = \frac{k+1}{2k} \left( \frac{(k+1)^2 - 1}{(k+1)^2} \right) = \frac{k+1}{2k} \left( \frac{k^2 + 2k}{(k+1)^2} \right) = \frac{k(k+2)}{2k(k+1)} = \frac{k+2}{2k+2} \)

BUT THIS IS \( P(k+1) \).

HENCE WE HAVE SHOWN \( P(k) \Rightarrow P(k+1) \), WHICH COMPLETES THE INDUCTION STEP AND THE PROOF.
3. \( b_0 = 1, \quad b_k = 2b_{k-1} + 3, \quad \forall k \geq 1 \).

a) \[
\begin{align*}
& b_1 = 5, \quad 8 = 2^3 \\
& b_2 = 13, \quad 16 = 2^4 \\
& b_3 = 29, \quad 32 = 2^5 \\
& b_4 = 61, \quad 64 = 2^6 \\
& b_5 = 125
\end{align*}
\]

b) \[
b_k = 5 + 2^3 + 2^4 + 2^5 + \ldots + 2^{k+1}
\]

c) \[
b_k = 5 + \sum_{m=0}^{k+1} 2^m - (1+2+2^2)
\]
\[
= 5 + \frac{2^{k+2} - 1}{2-1} - 7
\]
\[
= 2^{k+2} - 1 - 2
\]
\[
b_k = 2^{k+2} - 3, \quad \forall k \geq 0
\]

\[\Rightarrow 4 \cdot 2^{k+2} - 3\]

If guessed \( b_k = 2^{k+2} - 3, \quad \forall k \geq 0 \) use P(1). By induction to verify it is correct.

Base case \( k = 0 \). \( b_0 = 2^2 - 3 = 1 \) TRUE.

Let \( P(k) \) be statement \( b_k = 2^{k+2} - 3, \quad \forall k \geq 0 \)

Show \( P(k) \rightarrow P(k+1) \).

If \( P(k) \) TRUE THEN \( b_k = 2^{k+2} - 3 \). \( b_{k+1} = 2b_k + 3 \) by defn of seq.

\[
\Rightarrow b_{k+1} = 2(2^{k+2} - 3) + 3
\]

Subst.
\[
\Rightarrow b_{k+1} = 2^{k+3} - 6 + 3 \Rightarrow b_{k+1} = 2^{k+3} - 3 \Rightarrow P(k+1) \text{ QED.}.
\]
4. \( a_0 = 5, \ a_1 = 3 \)
\[ a_k = 6a_{k-1} - 9a_{k-2}, \ \forall k \geq 2 \]

a) \[ a_2 = 6 \cdot 3 - 9 \cdot 5 = 18 - 45 = -27 \]
\[ a_3 = 6(-27) - 9(3) = -189 \]
\[ a_4 = 6(-189) - 9(-27) = -891 \]

b) Characteristic equation is \( t^2 - 6t + 9 = 0 \) or \((t-3)^2 = 0 \Rightarrow t = 3\)

Only one real root so
\[ a_k = C \cdot 3^k + D \cdot k \cdot 3^k = 3^k (C + Dk) \]

\[ k = 0 \]
\[ a_0 = 5 = 3^0 (C + 0D) \text{ or } 5 = C \]

\[ k = 1 \]
\[ a_1 = 3 = 3^1 (5 + D) \Rightarrow 1 = 5 + D \Rightarrow D = -4 \]

Hence \[ a_k = 3^k (5 - 4k), \ \forall k \geq 0 \]

c) From part a) \[ a_4 = -891 \]
From part b) \[ a_4 = 3^4 (5 - 44) = 81(5 - 44) = 81(-39) = -891 \]

The two agree!