This assignment is based on the document below. Within the document there are a number of problems related to the material. I have split up the problems into four problem sets as described below:

1. Problems 1–15 on pp. 15–16 and the five problems in the margin on page 16.
2. Problems 1–20 on page 19 and the four problems in the margin on page 19.

I’ve also decided to divide the class up into three groups. Each group will work the problems in the problem sets as described by the table below. The assignment is due February 15th at the beginning of class.

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<tbody>
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<td>margin, #1–5</td>
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<td>margin, #6–12</td>
<td>#1, 2</td>
<td>#7–10</td>
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<td>#13–20</td>
<td>#3, 4, 5</td>
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<td>Alain Cousin</td>
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February 7, 2007
Preface

Calculus has always been a difficult subject to learn well. In the last half century alone there have been literally scores of calculus books published, each trying harder than the next to simplify the subject.

At the turn of the century it was popular to teach calculus by using so-called infinitesimals. This approach had the advantage of making the basic theory quite intuitive and easy to understand, but mathematicians lacked a rigorous definition of just what infinitesimals were, and so anyone advancing beyond the basics quickly became lost. Thus we were led to the $\varepsilon/\delta$ approach to calculus, an approach that, although totally precise and rigorous, was a disaster for students to learn and teachers to teach. Most recently $\varepsilon$'s and $\delta$'s have been shelved along with all other attempts at teaching the basics of calculus. Instead we have settled into teaching specific methods for applying calculus in specific situations. The problem here, of course, is that even though individual methods might be fairly easy to master, there exist very many, seemingly distinct, methods to be learned, and even then most courses leave students hopelessly short.

This active evolution in the teaching of calculus was always prompted by continual dissatisfaction with earlier approaches and had nothing to do with new mathematical insights into calculus itself. Indeed there were no such "new insights"; that is, there were none until just recently.

In the early 1960s the mathematician Abraham Robinson pioneered a body of work known as Nonstandard Analysis which makes precise and mathematically rigorous the intuitively pleasing concept of infinitesimal. Originally Robinson's field was reserved as an advanced graduate subject, but the ideas are both simple and important, and in recent years everyone from standard mathematicians to economists, physicists, and social scientists have been using his methods with stunning success.

A most natural place for Robinson's insight is as a next (and possibly final) point in the evolution of the teaching of calculus. We can now develop calculus using infinitesimals and enjoy all of their simplicity and intuitive power, yet at the same time work in a mathematically precise and rigorous atmosphere. This approach, although quite new, has been used at a number of universities with remarkable success.

This book presents a rigorous development of calculus using infinitesimals in the style of Robinson. It does not make any attempt to cover the assorted methods of calculus for applications, but rather it concentrates on theory, the area which previously was so difficult. We feel that with this new approach, basic theory is now quite accessible to students—even those who are interested in calculus solely for its applications. Indeed a knowledge of basic theory lets one dispense with learning many of the canned methods in favor of attacking problems directly and formulating one's own methods.

The only prerequisite assumed for this book is a good foundation in high school mathematics.

Our early chapters deal with the field of mathematical logic. Some of this is necessary for an understanding of infinitesimal calculus, but much of it is not. That which can be skipped is indicated in the text.

We received help from a great many people during the preparation of this manuscript, much of it from students taking preliminary versions of the course. Special thanks are also due to Professors Frank Wattenberg and David Schaffer, and to Mitchell Spector. Finally, Denise Borsuk typed countless versions of the text and displayed patience, good humor, and skill. To her we are most grateful.

J. H. Henle
E. M. Kleinberg
God made the integers, all else is the work of man.  
Leopold Kronecker (1823-1891)

God created infinity, and man, unable to understand infinity, had to invent finite sets.  
Gian-Carlo Rota (1932–)

The history of modern mathematics is to an astonishing degree the history of the calculus. The calculus was the first great achievement of mathematics since the Greeks and it dominated mathematical exploration for centuries. The questions it answered and the questions it raised lay at the heart of man’s understanding of not only geometry and number, but also space and time and mathematical truth. It began with the surprising unification of two rather different geometrical problems, and almost immediately its ideas bore fruit in dozens of seemingly unrelated areas. The methods it developed gave the physical sciences an impetus without parallel in history, for through them natural science was born, and without them physics could not have progressed much further than the mystical vortices of Descartes.

In the beginning there were two calculi, the differential and the integral. The first had been developed to determine the slopes of tangents to certain curves, the second to determine the areas of certain regions bounded by curves. Algebra, geometry, and trigonometry were simply insufficient to solve general problems of this sort, and prior to the late seventeenth century mathematicians could at best handle only special cases.

The general idea of the calculus, its fundamental theorem, and its first applications to the outstanding problems of mathematics and the natural sciences are due independently to Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716).

Their work was certainly built on foundations laid by others, but their penetrating insights represented what is easily the most significant mathematical breakthrough since the Greeks. Remarkably, the powerful
Or how about the problem of finding
the area of the plane region bounded
by the curve \( y = x^2 \) and the lines
\( y = 0, x = 1, \) and \( x = 2 \):

These problems are child’s play for
any student armed with elementary

The greatest contribution of New-
ton and Leibniz lies in their abstrac-
tion, organization, and notation. While
each made his discoveries independently,
there were later accusations that
Leibniz’s work was not original. The
debate was frequently bitter and the chief
result was that English mathematicians
patriotically refused to use Leibniz’s far
superior notation. This was possibly
responsible, in large measure, for
England’s loss of scientific leadership
during the Enlightenment.

Ironically, while supporters of
Newton and Leibniz argued bitterly
over the discovery of the calculus,
neither side knew that a third man
had made the discovery indepen-
dently, Seki Kōwa (1642-1708) in
Japan. The yenwi, or “circle method,”
an early form of the calculus, was
found at about the same time as
Newton and Leibniz, and, although
there is no direct evidence, it is most
likely that this was due to Seki Kōwa.

methods developed by these two men solved the same
class of problems and proved many of the same theo-
rems yet were based on different theories. Newton
thought in terms of limits whereas Leibniz thought in
terms of infinitesimals, and although Newton’s theory
was formalized long before Leibniz’s, it is far easier to
work with Leibniz’s techniques.

The approach to the calculus we shall employ is
based on Leibniz’s ideas as formalized by Abraham
Robinson in 1961 under the name of “nonstandard
analysis.” Simply stated, our approach will involve
expanding the real number system by introducing new
numbers called “infinitesimals.”

These new numbers will have the property that
although different from 0, each is smaller than every
positive real number and larger than every negative
real number. Of course our infinitesimals cannot them-
selves be real numbers, but what? This sort of ex-
pansion of a number system through the introduction
of new numbers which themselves correspond to noth-
ing in the real world is common in mathematics. Negati-
ve numbers and imaginary numbers have no direct
physical presence in the real world, yet both serve an
essential role in solving problems about the real world.
These new infinitesimals, once suitably defined, will
enable us to solve general problems of slopes of tan-
tents and areas of regions with extraordinary ease.

Here’s an example:

We shall find the slope of the tangent to \( y = x^2 \)
At the point \((1, 1)\).

How can we approach this problem? We know how
to find the slope of a line given two points on the line,
but here we are given only one point plus the informa-
tion that the line is tangent to \( y = x^2 \) at that point.
Our solution is simple: We let \( \mathbb{C} \) be an infinitesimal
(positive, say) and consider two points on the curve of
\( y = x^2 \) which are infinitely close to one another, \((1, 1)\)
and \((1 + \mathbb{C}, (1 + \mathbb{C})^2)\):

Later in his life, Newton became
quite suspicious and guarded—at
the encouragement of friends. Fear-
ing that Leibniz might steal his ideas,
he was once deliberately obscure.
One letter to Leibniz reads in its
entirety: “6, 2c, d, 6c, 13e, 2f, 7l, 3l,
9n, 40, 40, 2r, 4e, 9e, 12r, x.”

We can certainly find the slope of the line going
through these two points:

It is
\[
\frac{dy}{dx} = \frac{(1 + \mathbb{C})^2 - 1}{(1 + \mathbb{C})} = \frac{\mathbb{C}^2 + 2\mathbb{C}}{\mathbb{C}} = 2 + \mathbb{C}.
\]
Similarly, the slope of the chord going through (1, 1) and \((1 - \infty, (1 - \infty)^2)\) is \(-2 - \infty\).

As \(dx\) becomes smaller and smaller, the approximation improves, and the limit, as \(dx\) approaches 0, is the desired slope. To be precise, if \(f(x)\) is any function, we define the limit of \(f(x)\) as \(x\) approaches a, equals b
\[
\lim_{x \to a} f(x) = b
\]
by "for all \(\varepsilon > 0\), there is a \(\delta > 0\) such that whenever \(0 < |x - a| < \delta\), then \(|f(x) - b| < \varepsilon\)." To apply this, we first define a function \(m(dx) = dy/dx\). By calculations we find that, given \(dy, dx\) is
\[
2dx + (dx)^2. \text{ Thus,}
\]
\[
m(dx) = \frac{2dx + (dx)^2}{dx}
\]
Finally we prove that according to the definition of limit,
\[
\lim_{dx \to 0} m(dx) = 2,
\]
that is, for every \(\varepsilon > 0\), there is a \(\delta > 0\) such that whenever
\[
0 < |dx - 0| < \delta,
\]
then
\[
|2dx + (dx)^2| - 2 < \varepsilon.
\]
The proof goes as follows: Suppose \(\varepsilon > 0\) is given. Let (by guessing) \(\delta\) equal \(\varepsilon\). We can now check that this \(\delta\) works: If \(0 < |dx - 0| < \delta\), then
\[
|2dx + (dx)^2| - 2 = |2dx - 2| - 2 = |dx - 0| - 2 < \varepsilon.
\]
Notice that if \(dx\) actually equals 0, then \(dy = 0\) too and \(m(0) = 0/0\), which is undefined. The mystery is that although both the numerator \(dy\) and the denominator \(dx\) approach 0 as \(dx\) approaches 0, the fraction itself approaches 2. This limit Newton called the "ultimate ratio."

In Newton's words: "The ultimate ratios in which quantities vanish are not really the ratios of

The expansion of number systems has occurred often in the history of mathematics and has usually marked a major turning point. Our problem of desiring a new number, an infinitesimal, is entirely similar to that, says, of the early algebraists struggling to solve equations.

To these men the known number system consisted of 0, the positive rational numbers, and possibly a few irrationals like \(\sqrt{2}\), \(\sqrt{3}\), and \(x\). When faced with an equation such as \(2x + 10 = 6\), they recognized no possible solution. Later generations of mathematicians invented symbols to represent the solutions to such equations, solutions we would call negative numbers, but the inventors still denied their existence. They recognized them only as symbols which could be manipulated in equations, but which were not actually numbers.

The adoption of negative numbers was a slow process which was not completed for centuries. During this period they were tested thoroughly for consistency to see if they could be used without harm. Only when a number system was solidly constructed which contained both positive and negative numbers were they finally accepted. The most interesting point is that these numbers had a formal existence long before they became accepted. At first they were mere symbols, and then only after centuries of use did they become numbers.

A similar story can be told of the birth of complex numbers. As early as 1545, Cardan had formal symbols for them which he enjoyed manipulating, but which he regarded as fictitious. Soon everyone was using them, but again only in a formal way. The great mathematician of the eighteenth century, Leonard Euler, remarked of them: "... and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, which necessarily constitutes them imaginary or impossible..." Recognition and acceptance of the complex numbers as numbers wasn't actually final until the nineteenth century.

The history of algebra, in large part, the history of number. From the earliest conception of 1, 2, 3... our idea of number has grown slowly and painfully. At each step the growth was the result of a need. The need for numbers to represent the solution to equations such as \(2x + 10 = 6\) led to the negative numbers. The need for numbers to represent 10 + 4 and 1 + 3 led to the
rational numbers. The need for numbers to represent the diagonal of a square of side 1

and the circumference of a circle of radius 1

Many of the ideas of the calculus were present in the mind of this man, but without analytic geometry, without even an adequate system of numeration, he was unable to proceed further. His murder in 212 B.C. by a Roman soldier symbolized the end of creative mathematics for nearly 1,800 years.

Oddly enough, Archimedes knew this too. Archimedes' proofs, as he published and communicated them to Conon, used the "method of exhaustion," a limit-type proof. For centuries this method was held as a model for calculus proofs. It was one of the chief reasons seventeenth- and eighteenth-century mathematicians rejected infinitesimals and eventually made Newton's approach rigorous. It was only early in the twentieth century that a letter of Archimedes was found in which he expounded on his method, not of proving theorems, but of the calculus led to the infinitesimals. At each step expansion of the number system met with opposition, and at each step the new numbers were formally accepted long before they were given the status of numbers. As late as the 1880s, there was a distinguished mathematician, Leopold Kronecker, who philosophically disputed the existence of irrational numbers. At each step the numbers went through a period of experimentation and trial and were finally accepted only when some mathematician was able to develop a consistent system which contained both the old and the new numbers.

The same pattern holds for the infinitesimals. Even after their existence had been denied, they were in constant use as formal symbols. After 300 years of such usage, their existence finally became established when Robinson developed a system containing both infinitesimals and the real numbers. Robinson's system is now called the hyperreal number system, and using it he was able to completely justify the Leibniz approach to the calculus.

Often the new numbers which mathematicians invent shed light on the old numbers. For example, complex numbers were very useful in understanding real numbers. We will find that this is precisely the case with the hyperreals. They will be used exclusively for proving the theorems of calculus, theorems about real numbers. The power and beauty of this method, compared to the theory of limits, is sometimes astonishing. As Leibniz knew, the method of infinitesimals is the easy, natural way to attack these problems, while the theory of limits represents the lengths to which mathematicians were willing to go to avoid them.

Before actually launching ourselves into the details of the calculus, let us conclude this introductory section by considering an example of the second sort of physical problem solvable by calculus, namely the problem of finding the area of a plane figure. We might as
but of discovering them. This method is precisely the method of infinitesimals!

Although this letter was unknown to seventeenth-century mathematicians, they suspected that the Greeks were hiding something. To quote Leibniz: "these theorems they completed with reduction ad absurdum proofs by which they at the same time provided rigorous demonstration and also concealed their methods."

It is not likely that Archimedes was deliberately misleading; rather he felt the "method of exhaustion" was the only legitimate means for proving the theorem. Less honorable motives, however, may be assigned to later mathematicians:

I have omitted a number of things that might have made it [the geometry] clearer, but I did this intentionally, and would not have it otherwise. The only suggestions that have been made concerning changes in it are to render it clearer to readers, but most of these are so malicious that I am completely disgusted with them. René Descartes (1596–1650)

And it is said Newton himself had difficulty with Analytic Geometry!

Our approach to the problem is very simple: Since we know how to find areas of rectangles, we will simply approximate the area of $A$ by placing a great many thin rectangles over the region and adding up these areas. How thin should these rectangles be? Infinitesimally thin! We proceed as follows:

If we divide the area up into rectangles of thickness $h$, where $h$ is a real number, not an infinitesimal, we can use formulas from high school mathematics to show that the sum of the areas of the rectangles is $\frac{7}{3} + \frac{3h}{2} + \frac{h^2}{6}$.

Does this formula still hold if $h$ is $\odot$, an infinitesimal? As it happens, it does, and so to find out the desired area, we note that if $\odot$ is an infinitesimal, so are $\odot/2$ and $\odot^2/6$. Thus $\frac{7}{3} + \frac{3\odot}{2} + \frac{\odot^2}{6}$ is infinitely close to $7/3$. But since the area of $A$ must be a real number, we argue, as we did in the tangent example, that the area must be $7/3$.

There is one crucial point in our area calculation that bears repeating. We derived the formula for the approximation $\frac{7}{3} + \frac{3h}{2} + \frac{h^2}{6}$ for an actual real number, but we then assumed the formula was true even if $h$ were $\odot$, an infinitesimal. With this the proof was simple and easy. In the next chapter we will construct what we call the hyperreal number system by adding new numbers, infinitesimals, to the reals, and the most important part of our work will be to guarantee that formulas that work for reals also work for hyperreals, including infinitesimals.

To accomplish this, we will have to very good idea of what we mean by "formula," and this is where the techniques of mathematical logic will come in. Earlier mathematicians who attempted to build the hyperreals were defeated by this idea. There are many, many formulas, in fact, infinitely many, and they seem to be in complete disorder. It was only with the concept of a mathematical language that Robinson was able to bring order out of chaos. With this one key idea and its twin concept, mathematical structure, it will be relatively easy for us to construct the hyperreals. We will do it slowly and carefully, leaving no loose ends, and when we are done we can attack the problems of the calculus with directness and ease.
Ironically, Leibniz, in addition to his development of the calculus and his contributions to law, religion, philosophy, and diplomacy, also proposed another calculus, a *calculus ratiocinator*. It would consist, he imagined, of “a general method in which all truths of the reason would be reduced to a kind of calculation.” In this dream Leibniz anticipated by almost 200 years the birth of mathematical logic.

I have so many ideas that may perhaps be of some use in time if others more penetrating than I go deeply into them someday and join the beauty of their minds to the labor of mine.

Gottfried Wilhelm Leibniz (1646-1716)

The job before us is to build the hyperreal numbers. At first glance this does not seem so difficult; all we have to do is take the real numbers and add some infinitesimals. There is, however, a hitch. As mentioned in Chapter 1, we want something very special out of the hyperreals. We want every formula that is true in the reals to be true in the hyperreals. This will not be easy. There are infinitely many formulas, one more complicated than the next, and yet we must construct our system in only a few pages.

This is where mathematical logic comes in. Without it, mathematicians tried in vain for 300 years to construct the hyperreals. With it, Robinson was able to surmount the difficulties with astonishing ease.

Language

... the Symboles serve only to make men go faster about, as greater Wind to a Winde-mill.

Thomas Hobbes (1588-1679)

The key to our construction is to study first the language of the formulas. In this way we can organize the formulas and see how they arise. It sounds a little messy, but it isn’t. When we actually build the hyperreals, it will be quite smooth and natural.

There are many different so-called mathematical systems, and each system has an appropriate language. The best way to understand them is to look at a number of examples.
This symbol was originally chosen because the Latin word for "or," vel, begins with v.

There are many ways we could construct a language. We could use symbols different from \( \land, \lor, \rightarrow \); and we could use different meanings for symbols, etc. We have chosen this language because it is fairly easy to read and in standard use. As it happens, practically any reasonable language would work as well.

One example of a possible change in our language is this: Some people would suggest that "P or Q" should mean "either P or Q, but not both." This is a different kind of "or," and the symbol commonly used for it is \( \lor \). We could use this instead of \( \lor \), but it is less convenient, and we won't.

This definition of implication is difficult for some people to accept. The idea that a false statement "implies" any statement at all (if \( P \) is false, then \( P \rightarrow Q \) is true no matter what \( Q \) is) is often challenged. At a party the great mathematician and philosopher Bertrand Russell tried to explain this point to a particularly obtuse individual who finally agreed to accept it if Russell could show that \( 0 \times 1 \) is 1. Indeed Russell was the Pope. Russell reflected briefly and then argued: if \( 0 = 1 \), then \( 1 = 2 \). Since 1 and the Pope are two, 1 and the Pope are one. Q.E.D.

Using parentheses ( ) the connectives can be used to make longer statements. For example,

\[ N(\neg c, d) \rightarrow (N(c, g) \lor N(f, a)) \]

\[ (N(c, g) \lor N(f, a)) \rightarrow (N(a, d)) \]

Finally, another symbol is useful. The symbol \( \sim \) means "not." For example, \( \sim N(a, d) \lor \sim N(b, f) \) means "or". We can take any two statements \( P \) and \( Q \) and put them together to get \( P \lor Q \). This new statement is true if and only if both \( P \) and \( Q \) are true. For example,

\[ (N(c, g) \lor N(f, a)) \rightarrow (N(a, d)) \]

\[ (N(a, d)) \rightarrow (N(c, g) \lor N(f, a)) \]

EXERCISES

Are the following statements true or false?

1. \( N(d, e) \land \neg N(b, f) \).
2. \( N(c, e) \).
3. \( \sim N(c, e) \).
4. \( \sim N(c, e) \).
5. \( \sim (N(g, c) \land \sim N(g, f)) \).
6. \( \sim (\sim N(f, e) \land \sim N(g, b)) \).
7. \( (N(b, e) \lor \sim N(b, c)) \land (N(b, e) \land \sim N(b, c)) \).

We could actually get by without all the symbols \( \land, \lor, \rightarrow, \sim \), but our sentences would be much harder to read. For example, instead of

\[ P \rightarrow Q \]

we could say \( \neg P \lor Q \). These two sentences mean the same thing.
Similarly, instead of $\neg (P \land Q)$, we could say
$\neg (P \land \neg Q)$.

Thus (with a great deal of difficulty) we could use just two symbols, $\neg$ and $\lor$, instead of four.

**Exercises**

1. Convince yourself that $P \rightarrow Q$ and $\neg P \lor \neg Q$ mean the same thing.
2. Convince yourself that $P \land Q$ and $\neg (P \lor \neg Q)$ mean the same thing.

For one interested in the "algebra" of logical connectives, it is possible to define a single connective that can replace all the others. Let "$\Rightarrow$" be defined by "$P \Rightarrow Q$ is true if and only if both $P$ and $Q$ are false."

Using "$\Rightarrow$" we can replace $\neg$, $\lor$, $\land$, and $\rightarrow$. For example, $\neg P$ is true exactly when $P \Rightarrow Q$ is true. Similarly, $P \lor Q$ can be replaced by $(P \Rightarrow Q) \lor (P \Rightarrow Q)$.

**Exercises**

1. Find a way to replace $\land$ by $\Rightarrow$.
2. Find a way to replace $\neg$ by $\Rightarrow$.
3. Invent "$\times$".

The discovery of "$\Rightarrow$" was first made by an American, C. S. Peirce, in 1880, though he probably used a different symbol. This and much of Peirce's work remained unnoticed by mathematicians. Peirce explained this by saying "My damned brain has a kink in it that prevents me from thinking as other people do."

"$\Rightarrow$" was rediscovered by H. M. Sheffer in 1913.

Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.

Johann Wolfgang von Goethe

(1749-1832)

Thus our constants are OG, C, G, OC, ID, M, GM, MC, CA, LL, J, A, AM, I, MG, and O. In addition we will have variables $x$, $y$, and $z$, the same connectives and grammatical aids as before, $\land$, $\lor$, $\neg$, $\rightarrow$, $\Rightarrow$, and $\vee$.

For greater flexibility we can add variables $x$, $y$, and $z$ to our language. These enable us to talk about objects without having to specify just which object we have in mind. For example, $N(x, y)$ is read "$x$ is now to $a$" (which turns out to be true for some $x$ and false for others), and $N(x, a) \lor N(x, d)$ is read "$x$ is next to $a$ or $x$ is next to $d$," which is true for all $x$.

**Exercises**

For which areas $x$ are the following statements true?

11. $N(x, c)$.
12. $\neg N(x, a) \lor \neg N(x, c)$.
13. $N(x, d)$.
14. $N(x, a) \land N(x, e)$.
15. $\neg N(x, b) \land N(x, c)$.

Example 2. Let us study the genealogical table on page 17. We are interested in this time in three relationships: the two-place relation "is a parent of," the two-place relation "is married to," and the one-place relation "is a female."

Our language must include constants for all the people in the chart. Rather than use complete names, we will use the following abbreviations: OG for Owain Gwynedd, C for Christina, etc.

<table>
<thead>
<tr>
<th>C</th>
<th>OG</th>
</tr>
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<tbody>
<tr>
<td>M</td>
<td>G</td>
</tr>
<tr>
<td>CA</td>
<td>GC</td>
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<tr>
<td>LL</td>
<td>J</td>
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<tr>
<td>A</td>
<td>GM</td>
</tr>
<tr>
<td>MG</td>
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</tr>
</tbody>
</table>

Thus our constants are OG, C, G, OC, ID, M, GY, MC, CA, LL, J, A, GM, I, MG, and O. In addition we will have variables $x$, $y$, and $z$, the same connectives and grammatical aids as before, $\land$, $\lor$, $\neg$, $\rightarrow$, $\Rightarrow$, and $\vee$.

Three new relation symbols:

- $M(x, y)$—"$x$ is married to $y$"
- $P(x, y)$—"$x$ is a parent of $y$"
- $F(x)$—"$x$ is female"

**Third Generations of Lords of Wales**

All males trace their lineage to Meryn Frych (the freckled).

<table>
<thead>
<tr>
<th>Owain Gwynedd</th>
<th>Christina</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prince (tyranny) of Gwynedd, 1137–1170</td>
<td>Daughter of Gronw ab Owain ab Edwin</td>
</tr>
<tr>
<td>Gwenwynwyn—Margaret</td>
<td>Ioerthow</td>
</tr>
<tr>
<td>Lord of the Corbet</td>
<td>Derynwy</td>
</tr>
<tr>
<td>Owen Owain</td>
<td>Owain Cymyl</td>
</tr>
<tr>
<td>Owain Owain</td>
<td>Owain Owain</td>
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<td>Owain Owain</td>
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<td>Owain Owain</td>
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</table>

Gwenwynwyn—Margaret

[Lord of the Corbet

S. Powys

Caewallow

known as Wenwynwyn

—1195–1208,

deposed by his cousin

Llywelyn]

Angharad—Gruffydd Maelor I

[Lord of N. Powys

—1160–1191]

Isota—Maid of Gruffydd

Owain

[Lord of N. Powys

—known afterwards as Powys Fadog

—1151–1256]

All males are in italics.

Source: "Handbook of British Chronology," Sir F. Maurice Powicke and E. B. Fryde, ed. (Royal Historical Society, 1965). Note: The last independent Welsh prince was Llywelyn the Last, the greatnephew of Llywelyn ap Iorwerth.

To further enrich our language, we add the "$\Rightarrow$" sign and two quantifiers: $\forall$ and $\exists$.

You can think of "$\Rightarrow$" as another 2-place relation, but we write it in the usual way. For example, $x \Rightarrow GM$ instead of $= (x, GM)$, and $x \neq GM$ instead of $\neq$.
(x, GM). The quantifiers may also be familiar. ∀ means “for all,” and it is used with a variable. For example,

\[ \forall x \sim (M(x, O)) \]

is read “for all x, x is not married to Owain,” that is, Owain is unmarried, which is true. Another example:

\[ \forall y (F(G, y) \rightarrow \sim F(y)) \]

is read “for all y, if Gwenllian is a parent of y, then y is not female,” which is also true. \( \exists \) means “there exists,” and it is also used with a variable. For example,

\[ \exists x(P(x, ID, x)) \]

is read “there exists an x such that Ioerworth Drewydawn is a parent of x”—a true statement; and

\[ \exists z(P(z, J) \land P(z, O)) \]

is read “there exists a z such that z is a parent of Joan and a parent of Owain”—a false statement.

One last remark: Is the statement

\[ x = J \]

true or false? Neither, of course, since we don’t know what x is. For this reason we give the following definition.

**DEFINITION:** A statement in a language is said to be a *sentence* if every variable appearing in it

... x ...

is referred to by a quantifier

\[ \forall x(..., x ...) \]

or

\[ \exists x(..., x ...) \].

Thus, we could make “\( x = J \)” into a sentence by attaching a quantifier, getting either

\[ \forall x(x = J) \quad \text{(false)} \]

or

\[ \exists x(x = J) \quad \text{(true)}. \]

It should be clear that any sentence is always either true or false.

---

**EXERCISES**

In each statement discover who x is.

1. \( M(x, GY) \).
2. \( P(M, x) \).
3. \( P(C, x) \land \sim F(x) \).
4. \( P(GM, x) \land \forall y \sim M(y, x) \).
5. \( P(OC, x) \land \sim (x = GY) \).
6. \( \exists y[P(y, x) \land \exists zP(x, z) \land \sim F(x)] \).
7. \( F(x) \land \exists y[P(x, y) \land \forall zP(x, z) \rightarrow z = y] \).
8. \( \sim F(x) \land \exists y \exists z[P(x, y) \land P(y, z)] \).
9. \( F(x) \land \exists y[P(OC, y) \land P(y, x)] \).
10. \( \exists z[P(z, y, x) \land P(y, LL)] \).
11. \( \sim F(x) \land \exists y \exists z[P(x, y) \land P(x, z) \land [F(y) \rightarrow \sim F(y)] \land [F(z) \rightarrow \sim F(x)] \).
12. \( \exists y[P(x) \land M(y, x) \land P(y, x) \land \sim F(x)] \).

Allowing additional variables \( x_1, x_2, x_3, \ldots \), write statements defining the following relationships:

13. x is a brother of y. For example, \( \sim (F(x) \land \exists y[P(x, y, x) \land P(y, x)]) \) \( \land (x \neq y) \).
14. x is a grandfather of y.
15. x is a cousin of y.
16. x is a brother-in-law of y. (Careful—there are two different ways of being a brother-in-law.)
17. x is a niece of y. (This is also tricky.)
18. x is a stepmother of y.
19. x is a half brother of y.
20. x is a bastard.

**EXERCISES**

Define a new 2-place relation symbol \( L(x, y) \) meaning “x loves y.”

**EXERCISES**

1. Write in the language “Everybody loves somebody.”
2. Write in the language “All mankind loves a lover”—Emerson (interpret this as you will).

---

**Example 3.** This diagram represents pictorially a function \( F \). An arrow going from one point x to another point y means that the value of the function at x is y. For example, \( F(a) = b, F(d) = c, \) and \( F(f) = f \).
One of the earliest studies of logic was Aristotle’s, and it too involved the study of a special language. By comparison it was crude, allowing only statements of the form:

All are are.
No are are.
Some are .
Some are not .

From such sentences were formed arguments called syllogisms. Here is a typical syllogism:

All dogs are animals.

No fish are dogs.

All fish are animals.

The arguments were analyzed to discover whether they were valid or invalid. (The above is invalid.) Most of the work in logic for the next 2,000 years consisted of classifying and analyzing such arguments—which is less a tribute to Aristotle than a reflection on mathematics in the Middle Ages.

As late as the nineteenth century, mathematicians still looked for new ways to solve syllogisms. Robert Venn invented his Venn diagrams for just this purpose, (you may have studied them in high school). Charles Dodgson (author of Alice in Wonderland under the name Lewis Carroll) also invented a method which he published under the title The Game of Logic.

The term “function” (“Funktion” in German) was first coined by Leibniz.

Please note that the language described here is no more complicated than previous ones; it is simply bigger.

language will use constant symbols $a$, $b$, $c$, $d$, $e$, $f$, $g$, variables $x$, $y$, $z$, connectives $\land$, $\lor$, $\neg$, quantifiers $\forall$, $\exists$, and a function symbol $F$ standing for the function.

**Exercises**

State whether the sentence is true or false.
1. $\forall x \exists y (F(x) = y)$.
2. $\forall x \exists y (F(x) = x)$.
3. $\exists x \forall y (F(x) = y \land x = y)$.
4. $\forall x (F(x) = x \rightarrow (x = f \lor x = g))$.
5. $\forall x \exists y (F(x) = x \land F(y) = x \land \neg \neg (y = z)) \rightarrow (x = c \lor x = g)$.
6. $\exists x \exists y \exists z ((x = y) \land F(x) = y) \land (F(y) = z \land F(z) = x)$.
7. $\forall x (F(x) = F(F(x))) \rightarrow F(F(x)) = F(F(F(x)))$.

In each of our examples we have had the following categories:
1. constants
2. variables
3. grammatical symbols and connectives: $\land$, $\lor$, $\neg$, $\rightarrow$, (, )
4. quantifiers: $\forall$, $\exists$
5. relation symbols (including “= ”)
6. function symbols.

In the future our languages will be constructed from only these categories. All will have the first four categories and will generally have additional relation and function symbols. All our languages will have at least the relation symbol “=.”

Our next example is the most important one to us. It is a language to describe the real number system, and it is this language upon which our definition of the hyperreals will ultimately depend.

The formal construction of mathematical languages was accomplished as recently as the last century. Leibniz made some attempts that are, in retrospect, astonishingly modern, but they were ignored. Leibniz’s dream was that all thought could be written in such a language: “... when controversies arise, there will be no more necessity of disputation between two philosophers than between two accountants. Nothing will be needed but that they should take pen in hand, sit down with their counting tables and (having summoned a friend, if they like) say to one another: Let us calculate.”

Modern logic began with George Boole (1815–1864) who first experimented with the languages we are using here. Boole called it an algebra, because $\forall$ reminded him of addition, and $\land$ reminded him of multiplication. For example, $P \land Q$ and $Q \land P$ mean the same — $\land$ is “commutative.” Similarly, $P \land (Q \land R)$ and $(P \land Q) \land R$ mean the same — $\land$ is “associative.” Similarly, $\lor$ is both commutative and associative.

Finally, the distributive law which states $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ is also true in logic, that is, $P \land (Q \lor R)$ means the same as $(P \land Q) \lor (P \land R)$.

It is a good exercise to convince yourself of these facts.

To get the reader started on these exercises, the answers to the first two are:
1. $\forall x \exists y (x = y)$.
2. $\forall a \forall b (a = b \rightarrow a = b)$.

There are properties of the real numbers which cannot be expressed in $L$. Chief among these is the axiom of completeness.

**Exercises**

The following are most of the axioms for the real numbers, properties that we have seen many times before and accept without proof. Write them in the language $L$.

1. Every number is equal to itself.
2. If $a$ is equal to $b$, then $b$ is equal to $a$.
3. If $a$ and $b$ are both equal to $c$, then they are equal to each other.
4. Addition is commutative.
5. Addition is associative.
6. Zero is the additive identity.
7. Every number has an additive inverse.
8. Multiplication is commutative.
Definition: \( r \) is an upper bound for a set of reals \( B \) if \( r \) is greater than or equal to all elements of \( B \). \( r \) is the least upper bound of \( B \) if \( r \) is an upper bound, and no other upper bound for \( B \) is less than \( r \).

The Axiom of Completeness: For all nonempty sets of numbers \( B \), if \( B \) has an upper bound, then \( B \) has a least upper bound.

This axiom cannot be written in \( L \) because the phrase “for all nonempty sets of numbers” cannot be translated. We can quantify numbers in \( L \), but not sets of numbers.

9. Multiplication is associative.
10. 1 is the multiplicative identity.
11. Every number not equal to 0 has a multiplicative inverse.
12. Multiplication is distributive over addition.
13. The sum of two positive numbers is positive.
14. The product of two positive numbers is positive.
15. The additive inverse of a positive number is not positive.
16. Every number not equal to 0 is either positive or negative.

Structure

The essence of mathematics lies in its freedom.
Georg Cantor (1845–1918)

Example 5. Let us take another example of a language. We will describe a party of seven people represented by the constants \( a, b, c, d, e, f, \) and \( g \). We will have one relation symbol \( N(x, y) \), which means “\( x \) shook hands with \( y \).” In the above diagram persons who shook hands are identified by a line drawn between them.

Looking back to page 14, we see that this language is exactly the same as the one in example 1. The identical language is being used in two different contexts. Consider the sentence

\[ N(e, f). \]

Is it true? Well, it is true about example 5 but false about example 1. This shows us that the truth of a statement depends on the context. In mathematics this idea of a “context” is known as structure. So far in this chapter, we have had examples of four different languages and five different structures (or contexts).

Definition: A structure \((S, R, F)\) appropriate to a given language \( \mathcal{L} \) consists of three things: \( S \), a set of elements; \( R \), a set of relations on \( S \); and \( F \), a set of functions on \( S \); such that

1. Each constant of \( \mathcal{L} \) corresponds to some element of \( S \);
2. Each relation symbol of \( \mathcal{L} \) corresponds to some relation on \( S \) in \( R \); and
3. Each function symbol of \( \mathcal{L} \) corresponds to some function on \( S \) in \( F \).

In example 5 the set \( S \) was the group of people, there were no functions, and the relation was that in which people shook hands with each other.

Example 6. If the same seven people got together the next night and had a slightly different pattern of shaking hands,

we would have a different structure because, although the set \( S \) is the same, we would have a different relation.

We generally give the structure with sets \( S \), \( R \), and \( F \) the name \((S, R, F)\). \( S \) may have elements that do not correspond to constants, but every constant must correspond to some element of \( S \).

Example 7. Another structure appropriate to the language of examples 1 and 5 is the alphabet

\[ S = \{a, b, c, \ldots, x, y, z\} \]
\[ R = \{ \text{the alphabetical ordering} \} \]
\[ F = \{} \]

In this case \( N(e, f) \) is true because \( e \) comes before \( f \) in the alphabet.

Conversely, although all constants must correspond to elements of \( S \), they need not correspond to different elements.
The first coherent attempts to invent a comprehensive mathematical language were made independently by Gottlob Frege in 1879 and Giuseppe Peano in 1889. Later, Bertrand Russell and Alfred North Whitehead invented a language that was able to express virtually all known mathematical thought.

Example 8. Here is another structure appropriate to the same language:

\[ \begin{align*}
S &= \{17\} \\
R &= \{\text{the relation that nothing is related to itself}\} \\
F &= \{\}.
\end{align*} \]

(That is, \( N(17, 17) \) is false.) In this case all the constants, \( a, b, c, d, e, f, \) and \( g \) correspond to 17, the only element of \( S \), and \( N(e, f) \) is false since both \( e \) and \( f \) represent 17. All of the structures in examples 1, 5, 6, 7, and 8 are appropriate to the same language.

In example 2, \( S \) is a set of 16 people, \( F \) is the empty set, and \( R \) contains three relations, the parent-child relation, the female relation, and the marriage relation. In example 3, \( S \) is a set of 7 letters, \( F \) contains the function described by the arrows, and \( R \) is empty. In example 4, \( S \) is the set of all real numbers, \( F \) is the set of all functions on the reals, and \( R \) is the set of all relations on the reals.

One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we, wiser even than their discoverers, that we get more out of them than was originally put into them.

Heinrich Hertz (1857–1894)

The logic we are working with here is called 2-valued logic, because every sentence is either true or false. It is possible to invent more complex logics where sentences are either true, false, or partly true. We certainly don't need that here. Modern 3-valued logic was invented by Łukasiewicz in 1921. Oddly enough, it was discovered in 1936 that an English mathematician had also invented it—William of Occam (1270–1349)!

3

The Hyperreal Numbers

We are now in a position to state precisely what we mean by a "hyperreal number system."

**Definition.** A structure \( S \) is a hyperreal number system if it has the following three properties:

1. \( S \) contains the real number system. By this we mean not only that all real numbers are in \( S \), but also that every function and relation defined on reals is also defined on numbers in \( S \).
2. \( S \) contains an infinitesimal. That is, there is a number \( \varepsilon \) in \( S \) such that \( 0 < \varepsilon \) and yet \( \varepsilon < r \) for every positive real number.
3. The same sentences of \( L \) are true in both \( S \) and \( \mathbb{R} \). If \( B \) is any sentence of \( L \), then \( B \) is true in \( S \) if and only if \( B \) is true in \( \mathbb{R} \).

Notice that we defined what we mean by a hyperreal number system rather than *the* hyperreal number system. This is simply because there are many different hyperreal number systems. What is remarkable is that for the purpose of doing calculus any one of them is as good as any other.

In this chapter we shall construct a particular hyperreal number system denoted by \( HR \). (Henceforth, we'll call this particular system "the hyperreals.")

An understanding of the construction itself is not necessary for pursuing hyperreal calculus—one can fully understand the subsequent chapters of this book armed solely with a good comprehension of the above definition of a hyperreal number system.

At this point the reader may move immediately to Chapter 4, or continue with this chapter if he wishes