1. Write the negation of each of the following statements:

(a) “Jim is inside and Jan is at the pool.”

\[ p \land q \]

\[ \sim (p \land q) \equiv \sim p \lor \sim q \]

(b) “If it is Tuesday, then I’m in Belgium.”

\[ p \rightarrow q \]

\[ \sim (p \rightarrow q) \equiv p \land \sim q \]

2. Write the converse, inverse and contrapositive of the statement: “If it is Tuesday, then I’m in Belgium.” Which of these statements is logically equivalent to the original statement?

- **converse:** If I’m in Belgium, then it is Tuesday.
- **inverse:** If it is not Tuesday, then I’m not in Belgium.
- **contrapositive:** If I’m not in Belgium, then it is not Tuesday.

3. Is \( p \lor q \rightarrow p \equiv p \lor (\sim p \land q) \)? Complete the truth table below and explain your conclusion.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>\sim p</th>
<th>p \lor q</th>
<th>\sim p \land q</th>
<th>p \lor (\sim p \land q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
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<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

They are not logically equivalent since they don’t have the same truth values for all possible truth values for \( p \) and \( q \).

4. Consider the argument form:

\[ p \rightarrow \sim q \]

\[ q \rightarrow \sim p \]

\[ \therefore p \lor q \]

Use the truth table below to determine whether this is a valid argument form. Clearly explain how you made your decision.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>\sim p</th>
<th>\sim q</th>
<th>p \rightarrow \sim q</th>
<th>q \rightarrow \sim p</th>
<th>p \lor q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
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<td>T</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

It is not a valid argument since it is possible for both hypotheses to be true and the conclusion to be false (see the last row in the table).

5. Is the argument below logically valid? Justify your answer.

If Anne has the flu, then Anne has the fever.

Anne has the fever.

Therefore, Anne has the flu.

No. This argument is of the form: \( p \rightarrow q \). \[ q \]

\[ \therefore p \]

which is converse error.
No calculators are allowed on this quiz, so put them away now. Please show your work as completely as necessary to convince me that you know what you’re doing.

1. Each number below is the eight-bit binary representation of an integer $n$ in the range $-128 \leq n \leq 127$. Write the decimal and hexadecimal representation of each integer.

   (a) 01010011
   
   **Most significant bit “0”**
   
   So positive:
   
   \[64 + 16 + 2 + 1 = 83_{10}\]
   
   \[= 53_{16}\]

   (b) 11110010
   
   **Most sig. bit “1”**
   
   So negative.
   
   Plus comp is: 00001110 + 1
   
   \[= 00001110\]
   
   \[= (8 + 4 + 2)_{10} = 14_{10}\] so
   
   \[-14_{10}\]

2. Find the sum 10111010 + 01110111, where both numbers are eight-bit binary representations of integers in the interval $[-128, 127]$. You must use binary arithmetic to perform the sum and give the answer in binary form.

   
   \[
   \begin{array}{c}
   10111010 \\
   + 01110111 \\
   \hline
   100110001
   \end{array}
   \]

3. Let $P(n)$ be the predicate "$n$ is even".

   (a) What is the truth set of $P$ if the domain of $n$ is the positive integers?
   
   Truth set is \[2, 4, 6, 8, \ldots \, 3\]

   (b) Write the statement "\(\forall x \in \mathbb{Z}, P(x^2) \rightarrow P(x)\)" in an informal way.

   "If the square of any integer is even, then the integer is even."

4. Rewrite the statement "Some people like mathematics.", in the forms:

   (a) \(\exists \) ______ $x$ such that ______.
   
   \(\exists\) **A person** $x$ such that *$x$ likes mathematics.*

   (b) \(\exists x\) such that ______ and ______.

   \(\exists x\) such that *$x$ is a person and $x$ likes mathematics.*
No calculators are allowed on this quiz, so put them away now. Please show your work as completely as necessary to convince me that you know what you’re doing.

1. Circle all of the statements below which are logically equivalent to the NEGATION of the statement “All dogs are loyal.”

   (a) All dogs are disloyal.  
   (b) No dogs are loyal.  
   (c) Some dogs are disloyal.  
   (d) Some dogs are loyal.  
   (e) There is a disloyal animal that is not a dog.  
   (f) There is a dog that is disloyal.  
   (g) No animals that are not dogs are loyal.  
   (h) Some animals that are not dogs are loyal.

2. Given the statement “All the cats in this class are yellow.”

   (a) Write this statement formally using the form: \( \forall x \in C, P(x) \rightarrow Q(x) \), where \( C \) is the set of all things in class. Write out in sentence form the predicates \( P(x) \) and \( Q(x) \).

   \[ P(x) = \text{"x is a cat"} \]
   \[ Q(x) = \text{"x is yellow"} \]
   \[ \forall x \in C, (P(x) \rightarrow Q(x)) \]

   (b) Write the formal negation of this statement and use its truth value to determine the truth value of the original statement.

   \[ \neg \exists x \in C \text{ s.t. } (P(x) \rightarrow Q(x)) \]
   \[ \rightarrow \text{ This is false (there is no cat in class) so original statement is true.} \]

3. Write the statement “Being on time each day is a necessary condition for keeping this job” in if-then form.

   “If you are not on time each day then you will not keep this job.”
   
   or
   
   “If you keep this job then you will be on time each day.”

4. Let \( D = E = \{-2, -1, 0, 1, 2\} \). Write negations for each of the following statements and determine which is true, the given statement or its negation.

   (a) \( \forall x \in D, \exists y \in E, x + y = 1 \)

   \[ \neg \text{NEGATION: } \exists x \in D, \forall y \in E, x + y \neq 1 \]
   \[ \neg \text{NEGATION is TRUE \( x = -2 \)} \]

   (b) \( \exists x \in D, \forall y \in E, x + y = -y \)

   \[ \neg \text{NEGATION: } \forall x \in D, \exists y \in E, x + y \neq -y \]
   \[ \neg \text{NEGATION is TRUE} \]
1. Consider the statement: “For all integers \( m \) and \( n \), if \( 2m + n \) is odd then \( m \) and \( n \) are both odd.”

(a) Write the negation of the statement.

\[
\exists m, n \in \mathbb{Z} \text{ s.t. } (2m + n \text{ odd}) \land \left[ (m \text{ even}) \lor (n \text{ even}) \right]
\]

(b) Show that the original statement is false by showing its negation to be true.

\[
\begin{align*}
\text{Let } & \quad m = 2, \quad n = 1 \\
\text{Then } & \quad 2m + n = 2 \cdot 2 + 1 = 5 \text{ is odd} \\
\text{and } & \quad m = 2 \text{ is even.}
\end{align*}
\]

2. If \( m \) and \( n \) are integers, is \( 6m^2 + 34n - 18 \) an even integer? Justify your answer.

\[
6m^2 + 34n - 18 = 2(3m^2 + 17n - 9) = 2k, \quad \text{with } k = 3m^2 + 17n - 9 \in \mathbb{Z}
\]

(by closure axioms of \( \mathbb{Z} \))

Hence \( 6m^2 + 34n - 18 \) is an even integer.

3. Is the number \( 3.1\overline{4} \) a rational number? Justify your answer using the definition of rational number.

\[
\begin{align*}
\text{Let } & \quad x = 3.1\overline{4} = 3.1444.... \\
\text{Then } & \quad 10x = 31.\overline{4}, \quad 100x = 314.\overline{4} \\
\text{Hence } & \quad 100x - 10x = 90x = 314.\overline{4} - 31.\overline{4} = 283 \\
or & \quad x = \frac{283}{90} \in \mathbb{Q}
\end{align*}
\]

4. Find a counterexample to show that the statement below is false

\[
\forall a, b, c, d \in \mathbb{R}^*, \quad \frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}
\]

where \( \mathbb{R}^* \) is the set of non-zero real numbers.

Consider \( a = b = c = d = 1 \)

\[
\frac{a}{b} + \frac{c}{d} = 1 + 1 = 2
\]

\[
\frac{a + c}{b + d} = \frac{2}{2} = 1
\]

Hence \( \frac{a}{b} + \frac{c}{d} \neq \frac{a + c}{b + d} \forall a, b, c, d \in \mathbb{R}^* \)

February 22, 2007
1. If \( n = 4k + 1, k \in \mathbb{Z} \), does \( 8 \mid (n^2 - 1) \)? Explain.

\[
\begin{align*}
n^2 - 1 &= (4k+1)^2 - 1 = 16k^2 + 8k + 1 - 1 = 16k^2 + 8k = 8(2k^2 + k) \\
\text{Hence } n^2 - 1 &= 8m, \text{ with } m = 2k^2 + k \in \mathbb{Z} \text{ (by closure axioms of } \mathbb{Z}) \\
\Rightarrow & \quad 8 \mid (n^2 - 1) \text{ by defn of divisibility.}
\end{align*}
\]

2. Prove: \( \forall a, b, c \in \mathbb{Z}, \quad (a|b) \land (b|c) \rightarrow a|(b + c) \).

\text{Showed } (a|b) \land (b|c) \Rightarrow a|c \text{ in class.}

\text{Let } a, b, c \in \mathbb{Z} \text{ be particular but arbitrary integers with } (a|b) \land (b|c) \Rightarrow a|c \Rightarrow \exists m \in \mathbb{Z} \text{ s.t. } c = ma.

\text{Also have } a|b \Rightarrow \exists n \in \mathbb{Z} \text{ s.t. } b = na.

\Rightarrow b + c = ma + na = a(m + n) = ab, \text{ where } k = m + n \in \mathbb{Z}

\Rightarrow a|(b + c). \quad \text{QED.}

3. True or false? If true give a direct proof; if false give a counter-example.
\( \forall a, b, c \in \mathbb{Z}, \quad a|bc \rightarrow (a|b) \lor (a|c) \).

\text{False. Consider } a = 4, b = c = 2.

\text{Then } 4|1(2 \cdot 2) \text{ true but } 4|2 \text{ false.}

4. What are the values of 37 \( \text{DIV} \) 7 and 37 \( \text{MOD} \) 7?

\[
\begin{align*}
37 \text{ div } 7 &= 5 \\
37 \text{ mod } 7 &= 2
\end{align*}
\]

5. When an integer “a” is divided by 7 the remainder is 4. What is the remainder when “5a” is divided by 7?

\[a = 7q + 4 \text{ for some } q \in \mathbb{Z} \text{ (by QUOTIENT-REMAINDER THM)}\]

\[\Rightarrow 5a = 35q + 20 = 35q + 14 + 6 = 7(5q + 2) + 6\]

\text{or } 5a = 7q + 6, \text{ where } q = 5q + 2 \in \mathbb{Z}

\Rightarrow \text{remainder is } 6.\]
Give complete, clearly stated proofs below. You might find the identity \((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\) helpful.

1. Consider the proposition: \(\forall n \in \mathbb{Z}\) if \(n^3\) is even then \(n\) is even.

   (a) Write the negation of this proposition.
   \(\exists n \in \mathbb{Z} \text{ s.t. } n^3 \text{ even and } n \text{ is odd.}\)

   (b) Write the contrapositive of this proposition.
   \(\forall n \in \mathbb{Z} \text{ if } n \text{ is odd then } n^3 \text{ is odd.}\)

   (c) Prove the proposition by the method of contradiction.
   \begin{itemize}
   \item Suppose prop. is not true.
   \item Then \(\exists n \in \mathbb{Z} \text{ s.t. } n^3 \text{ even and } n \text{ is odd.}\)
   \item Let \(n \in \mathbb{Z}\) with \(n\) odd and \(n^3\) even.
   \item \(\text{Let } n = 2k+1, \text{ s.t. } \begin{align*}
   n^3 &= (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 \\
                 &= 2(4k^3 + 6k^2 + 3k + 1) + 1 \\
   \text{ or } n^3 &= 2m+1 \text{ with } m = 4k^3 + 6k^2 + 3k \in \mathbb{Z} \quad (\text{by closure axioms})
   \end{align*}
   \item \(\Rightarrow n^3 \text{ is odd, but } n^3 \text{ even by hypothesis.}\)
   \item Hence proposition must be true.
   \end{itemize}

   (d) Prove the proposition by contraposition.

   \begin{itemize}
   \item The proposition is logically equivalent to its contrapositive:
   \(\forall n \in \mathbb{Z}, \ n \text{ odd } \rightarrow n^3 \text{ odd}\)
   \item \(\text{Prop.: let } n \text{ be a particular but arbitrarily chosen odd integer}\)
   \item Then \(n = 2k+1 \text{ for some } k \in \mathbb{Z}\)
   \item Hence \(n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1\)
   \item So \(n^3 = 2m+1, \text{ with } m = 4k^3 + 6k^2 + 3k \in \mathbb{Z} \quad (\text{by closure axioms})\)
   \item Hence \(n^3 \text{ is odd } \Rightarrow \text{QED.}\)
   \end{itemize}

March 8, 2007

P.1
2. Fill in the blanks of the following proof by contradiction that $7 + 4\sqrt{2}$ is an irrational number. You may use the fact, proved in class, that $\sqrt{2}$ is irrational.

**Proof:** Suppose not. Suppose that $7 + 4\sqrt{2}$ is _RATIONAL_. By definition of rational, $7 + 4\sqrt{2} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. Multiplying both sides by $b$ gives

$$7b + 4b\sqrt{2} = a$$

so if we subtract $7b$ from both sides we have

$$4b\sqrt{2} = a - 7b$$

Dividing both sides by $4b$ gives

$$\sqrt{2} = \frac{a - 7b}{4b}$$

By closure axioms and $4b \neq 0$ (since $b \neq 0$)

But then $\sqrt{2}$ would be a rational number because $a - 7b, 4b \in \mathbb{Z}$. This contradicts our knowledge that $\sqrt{2}$ is irrational. Hence $7 + 4\sqrt{2}$ _is IRRATIONAL_.

March 8, 2007
1. Compute \[ \sum_{k=1}^{4} \left( \frac{1}{k} - \frac{1}{k+1} \right) \]
\[ = (1 - \frac{1}{2}) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) \]
\[ = 1 - \frac{1}{5} = \frac{4}{5} \]

2. Use summation notation to write \[ -\frac{2}{1} + \frac{5}{4} - \frac{10}{9} + \frac{17}{16} - \frac{26}{25} + \frac{37}{36} \]

3. Compute \[ \prod_{n=5}^{51} \left( \frac{n+1}{n} \right) \]. Hint: Write out first three and last three factors in the product, look for a way to simplify the product.
\[ = \left( \frac{6}{5} \right) \left( \frac{7}{6} \right) \left( \frac{8}{7} \right) \cdots \left( \frac{50}{49} \right) \left( \frac{51}{50} \right) \left( \frac{52}{51} \right) \]
\[ = \frac{52}{5} \]

4. Use proof by mathematical induction to prove:

For any real number \( r, r \neq 1 \), \[ \sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r-1}, \quad \forall n \geq 0 \]

(a) Base case \( (n=0) \): \[ \sum_{k=0}^{0} r^k = \frac{r^1 - 1}{r-1} = 1 \Rightarrow r^0 = 1 \quad \text{True} \]

(b) Inductive hypothesis: Must show \( \forall n \geq 0 \) \( P(n) \rightarrow P(n+1) \) where \( P(n) \) is \[ \sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r-1} \]

Let \( n+1 \) with \( P(n) \) true, i.e.
\[ \sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r-1} \]

\[ 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r-1} \]

(Add \( r^{n+1} \) to both sides)
\[ 1 + r + r^2 + \cdots + r^n + r^{n+1} = \frac{r^{n+2} - 1}{r-1} + \frac{r^{n+1}}{r-1} \]

or
\[ 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r-1} + \frac{r^{n+1}}{r-1} \]

But this is \( P(n+1) \)! Q.E.D.
1. Find the first four terms of the sequence defined recursively by:
   \[ u_1 = 1; u_2 = 2; u_k = ku_{k-1} - u_{k-2}, \quad \forall k \in \mathbb{Z}, k \geq 3. \]
   
   \begin{align*}
   u_1 &= 1 \\
   u_2 &= 2 \\
   u_3 &= 3 \cdot 2 - 1 = 5 \\
   u_4 &= 4 \cdot 5 - 2 = 18 \\
   
   \end{align*}

2. Prove that \( F_{k+1}^2 - F_k^2 = F_k F_{k+2}, \) \( \forall k \geq 1, \) where \( F_k \) is the Fibonacci sequence. Hint: Start by factoring the left-hand side.

   \[
   \frac{F_{k+1}^2 - F_k^2}{F_k} = (F_{k+1} - F_k)(F_{k+1} + F_k)
   \]

   \[
   = (F_{k+1} + F_{k-1} - F_k)(F_{k+2})
   \]

   \[
   \text{Since: } F_{k+1} + F_k = F_{k+2}
   \]

   \[
   = F_{k+1} F_{k+2}
   \]

3. Consider the sequence defined recursively by
   \[
   g_1 = 1 \\
   g_k = \frac{g_{k-1}}{g_{k-1} + 2}, \quad \forall k \geq 2
   \]

   Use iteration to guess an explicit formula for the sequence. Hint: Consider powers of 2.

   \[
   \begin{align*}
   g_1 &= 1 = \frac{1}{2^1 - 1} \\
   g_2 &= \frac{1}{3} = \frac{1}{2^2 - 1} \\
   g_3 &= \frac{\left(\frac{1}{2}\right)^3}{\left(\frac{1}{2} + 2\right)^3} = \frac{1}{1 + 6} = \frac{1}{7} = \frac{1}{2^3 - 1} \\
   g_4 &= \frac{\left(\frac{1}{2}\right)^3}{\left(\frac{1}{2} + 2\right)^7} = \frac{1}{15} = \frac{1}{2^5 - 1} \\
   g_5 &= \frac{\left(\frac{1}{15}\right)_{15}}{\left(\frac{1}{15} + 2\right)_{15}} = \frac{1}{31} = \frac{1}{2^{15} - 1}
   \end{align*}
   \]

   Guess:

   \[
   \frac{1}{2^k - 1}
   \]

April 12, 2007
4. Consider the second order, linear, homogeneous recurrence relation with constant coefficients defined below:

\[ a_0 = 1, \ a_1 = 2 \]
\[ a_k = 2a_{k-1} + 3a_{k-2}, \ \forall k \geq 2 \]

(a) Write out the first five terms of the sequence.

(b) Solve the sequence; find a formula for \( a_k \) as a function of \( k \).

(c) Use the formula to find \( a_4 \) and verify that it agrees with your answer for \( a_4 \) from the first part of this problem.

\[
\begin{align*}
\text{a}) \quad a_0 &= 1 \\
a_1 &= 2 \\
a_2 &= 2 \cdot 2 + 3 \cdot 1 = 7 \\
a_3 &= 2 \cdot 7 + 3 \cdot 2 = 20 \\
a_4 &= 2 \cdot 20 + 3 \cdot 7 = 61
\end{align*}
\]

\[
\text{b}) \quad \text{CHAR EQU IS} \quad t^2 - 2t - 3 = 0 \implies (t-3)(t+1) = 0 \\
\implies t = 3 \quad \text{or} \quad t = -1 \\
\text{GEN SOLN IS} \quad a_k = C \cdot 3^k + D(-1)^k
\]

\[
\begin{align*}
\text{for} \quad k = 0 & \quad \implies 1 = a_0 = C + D \\
\text{for} \quad k = 1 & \quad \implies 2 = a_1 = 3C - D
\end{align*}
\]

\[
\begin{align*}
l = C + D \\
3 = 4C \\
\implies C = \frac{3}{4}
\end{align*}
\]

\[
\begin{align*}
C + D = 1 \\
\implies \frac{3}{4} + D = 1 \\
\implies D = \frac{1}{4}
\end{align*}
\]

\[ \text{HENCE:} \quad a_k = \frac{3}{4} \cdot 3^k + \frac{1}{4} (-1)^k \quad \text{or} \quad a_k = \frac{1}{4} (3^{k+1} + (-1)^k) \]

\[
\begin{align*}
\text{for} \quad k = 4 & \quad \implies a_4 = \frac{1}{4} (3^5 + (-1)^4) = \frac{1}{4} (243 + 1) = \frac{1}{4} (244) = 61
\end{align*}
\]

\[ \text{WHICH AGREES WITH WHAT WE GET BY ITERATION} \]
Find the quotient automaton $\bar{A}$ of the automaton $A$ shown below. Do this by finding equivalence classes of the equivalence relations $R_0$, $R_1$, $R_2$ and $R_3$, then identifying the equivalence classes of $R_4$ and using these equivalence classes as states of the quotient automaton.

![Automaton diagram]

Give the equivalence classes of each equivalence relation below.

1. $R_0: \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$
2. $R_1: \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$
3. $R_2: \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$
4. $R_3: \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$
5. $R_4: \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$

![Quotient automaton diagram]

$\bar{A}$ isomorphic to $A$.