RIGID BODIES

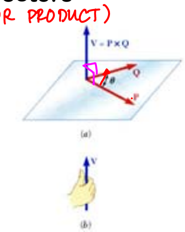
Cross Product of Two Vectors

(OR AKA VECTOR PRODUCT)

Given: $\vec{P}$ & $\vec{Q}$

Find: $\vec{P} \times \vec{Q}$

$\vec{V} = \vec{P} \times \vec{Q}$



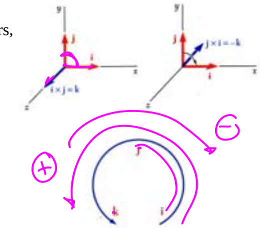
Geometric Definition:

- Vector product of two vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as the vector $\boldsymbol{V}$ which satisfies the following conditions:
  1. Line of action of $\boldsymbol{V}$ is perpendicular to plane containing $\boldsymbol{P}$ and $\boldsymbol{Q}$.
  2. Magnitude of $\boldsymbol{V}$ is $V = PQ\sin\theta$
  3. Direction of $\boldsymbol{V}$ is obtained from the right-hand rule.

- Properties of Cross products:
  - are not commutative, $\boldsymbol{Q}\times\boldsymbol{P} = -(\boldsymbol{P}\times\boldsymbol{Q})$
  - are distributive, $\boldsymbol{P}\times(\boldsymbol{Q}_1+\boldsymbol{Q}_2) = \boldsymbol{P}\times\boldsymbol{Q}_1 + \boldsymbol{P}\times\boldsymbol{Q}_2$ ✱
  - are not associative, $(\boldsymbol{P}\times\boldsymbol{Q})\times\boldsymbol{S} \neq \boldsymbol{P}\times(\boldsymbol{Q}\times\boldsymbol{S})$

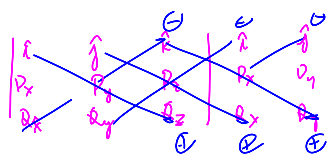
- Vector products of Cartesian unit vectors,

$$\vec{i}\times\vec{i}=0 \quad \vec{j}\times\vec{i}=-\vec{k} \quad \vec{k}\times\vec{i}=\vec{j}$$
$$\vec{i}\times\vec{j}=\vec{k} \quad \vec{j}\times\vec{j}=0 \quad \vec{k}\times\vec{j}=-\vec{i}$$
$$\vec{i}\times\vec{k}=-\vec{j} \quad \vec{j}\times\vec{k}=\vec{i} \quad \vec{k}\times\vec{k}=0$$



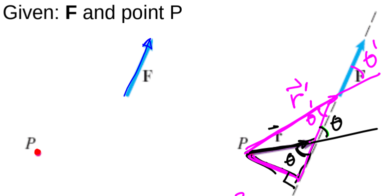
In terms of Components:

$$\vec{P}\times\vec{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} = (P_yQ_z - Q_yP_z)\hat{i} - (P_xQ_z - Q_xP_z)\hat{j} + (P_xQ_y - Q_xP_y)\hat{k}$$



The Moment Vector (AKA TORQUE)

Given: **F** and point P



$\vec{M}_P = \vec{r}\times\vec{F}$

$|\vec{M}_P| = |\vec{r}|\,|\vec{F}|\sin\theta$

$= |\vec{r}|\,|\vec{F}|\sin\theta$

$= |\vec{r}'|\,|\vec{F}|\sin\theta'$

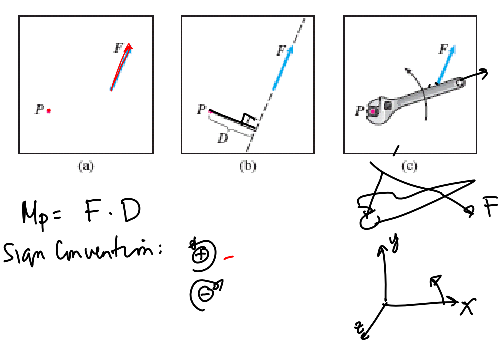
$|\vec{F}|\,\underbrace{|\vec{r}'|\sin\theta'}_{D_\perp}$

$M_P = F \cdot D_\perp$

$\vec{M}_P = \vec{r}\times\vec{F} \overset{?}{=} \vec{r}'\times\vec{F}$

$\vec{r}$ = position vector from P to any point in the line of action of $\vec{F}$.

Two-Dimensional Description of the Moment RHR

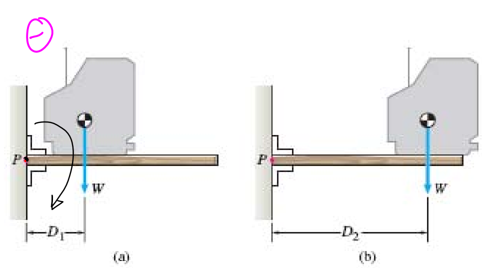


$M_P = F \cdot D$

Sign Convention: ⊕ −

⊖

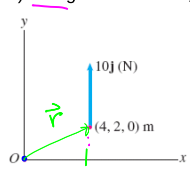
Moment of a force about a point is a measure of the tendency of a force to cause rotation about the point.



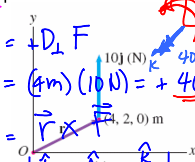
Relationship between 2-D Moment and the Moment Vector

Example. Find the moment of the force about the origin

a) Using 2-D definition, b) Cross product.



$\vec{r} = 4\hat{i} + 2\hat{j} + 0\hat{k}$



a) $M_o = +D_\perp F$

$= (4\,m)(10\,N) = +40\ N\text{-}m$

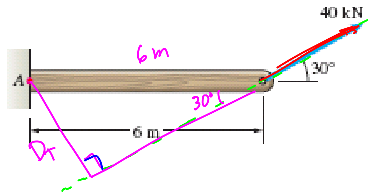
b) $\vec{M}_o = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4\,m & 2\,m & 0\,m \\ 0\,N & 10\,N & 0\,N \end{vmatrix}$$

$= (0-0)\hat{i} - (0-0)\hat{j} + (40-0)\hat{k}$

$\vec{M} = 40\,\hat{k}\ (N\text{-}m)$

Example: Find the moment of the 40 kN force about point A.

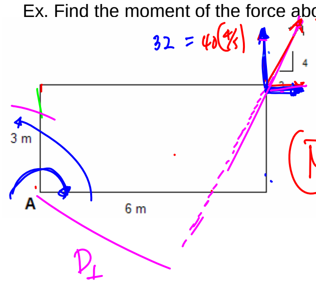


$M_A = +D_\perp F$

$= (6\,m \sin 30^\circ)(40\,kN)$

$\boxed{M_A = 120\ N\text{-}m}$

Ex. Find the moment of the force about point A.



$32 = 40(4/5)$

$F = \vec{F}_x + \vec{F}_y$

$40(3/5) = 24$

$(\vec{M}_{\vec{F}})_A = (\vec{M}_{\vec{F}_x})_A + (\vec{M}_{\vec{F}_y})_A$

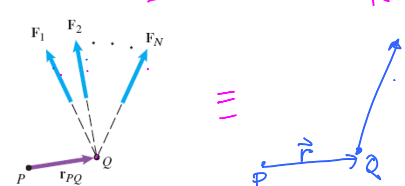
$= -(24)(3) + 32(6)$

$= -72 + 192$

$= 120\ kN\text{-}m$

Varignon's Theorem

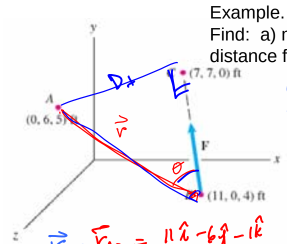
$\vec{R} = \sum_{i=1}^{N} \vec{F}_i$



$\sum (M_{\vec{F}})_P = (M_{\vec{R}})_P$

Example. The force **F** is of magnitude 90 lb.

Find: a) moment about A, b) perpendicular distance from A to the line of action of **F**.



Given: $|\vec{F}| = 90$

Find: $\vec{M}_A$

$\vec{M}_A = \vec{r} \times \vec{F}$

$\vec{F} = 90\,\hat{e}_{BC}$

$= 90\left(\dfrac{-4\hat{i} + 7\hat{j} - 4\hat{k}}{\sqrt{4^2+7^2+4^2}}\right)$

$\vec{F} = -40\hat{i} + 70\hat{j} - 40\hat{k}$

$\vec{r} = \vec{r}_{AB} = 11\hat{i} - 6\hat{j} - 1\hat{k}$

a) $\vec{M}_A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 11 & -6 & -1 \\ -40 & 70 & -40 \end{vmatrix} = 310\hat{i} + 480\hat{j} + 530\hat{k}\ (lb\text{-}ft)$

b) $r \sin\theta$

$|\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}| \sin\theta$

$\dfrac{|\vec{r} \times \vec{F}|}{|\vec{F}|} = |\vec{r}| \sin\theta \to D_\perp$

$\dfrac{\sqrt{(310)^2 + (480)^2 + (530)^2}}{90} = D_\perp$

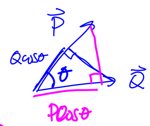
$8.66\ ft =$

(SCALAR)
Dot Product of Two Vectors

Given: $\vec{P}$ & $\vec{Q}$

Find: $\vec{P}\cdot\vec{Q}$ → scalar:

Geometric Definition:



$$\vec{P}\cdot\vec{Q} = P\,Q\cos\theta$$
$$= Q\,P\cos\theta$$

$\cos\theta$: ⊕ $-90° < \theta < 90°$; ⊖ $90° < \theta < 270°$

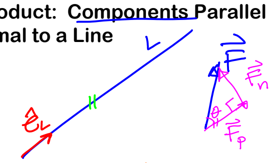
In terms of Components

$$\vec{P}\cdot\vec{Q} = P_xQ_x + P_yQ_y + P_zQ_z$$

Application of the Dot Product: Components Parallel and Normal to a Line

Given: $\vec{F}$, L ($\hat{e}_L$)

Find: $\vec{F}_p$ & $\vec{F}_n$



$$F_p = F\cos\theta = \hat{e}_L\cdot\vec{F}$$

$$\hat{e}_L\cdot\vec{F} = |\hat{e}_L|\,|\vec{F}|\cos\theta = 1\ F\cos\theta$$

$$\vec{F}_p = F_p\,\hat{e}_p$$

$$\boxed{\vec{F}_p = \underbrace{(\hat{e}_L\cdot\vec{F})}_{\text{magn.}}\underbrace{\hat{e}_L}_{\text{dir.}}}$$

$\vec{F}_n = ?$

$$\boxed{\vec{F}_n = \vec{P} - \vec{F}_p}$$

H. Mixed Triple Product

Given: $\vec{U}, \vec{V}, \vec{W}$ Find: $\vec{U}\cdot(\vec{V}\times\vec{W})$ scalar

$$\vec{U}\cdot(\vec{V}\times\vec{W}) = (U_x\hat{i} + U_y\hat{j} + U_z\hat{k})\cdot\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ V_x & V_y & V_z\\ W_x & W_y & W_z\end{vmatrix}$$

$$= (U_x\hat{i} + U_y\hat{j} + U_z\hat{k})\cdot\{(V_yW_z - W_yV_z)\hat{i} - (V_xW_z - W_xV_z)\hat{j} + (V_xW_y - W_xV_y)\hat{k}\}$$

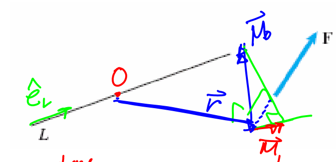
$$= U_x(V_yW_z - W_yV_z) - U_y(V_xW_z - W_xV_z) + U_z(V_xW_y - W_xV_y)$$

$$\boxed{\vec{U}\cdot(\vec{V}\times\vec{W}) = \begin{vmatrix}U_x & U_y & U_z\\ V_x & V_y & V_z\\ W_x & W_y & W_z\end{vmatrix}}$$

Moment of a Force about a Line

Given:

Force **F**, and line L



$\vec{M}_L$:

① Select a point O, anywhere in the line

② Find $\vec{M}$ about point O.

a) $\vec{r}$ from O to anywhere in $\vec{F}$

b) $\vec{M}_o = \vec{r}\times\vec{F}$

③ $\vec{M}_L$ is the component of $\vec{M}_o$ along the line L; $\vec{M}_L \parallel$ to $\hat{e}_L$

$$\boxed{\vec{M}_L = (\hat{e}_L\cdot\vec{M}_o)\hat{e}_L} = [\hat{e}_L\cdot(\vec{r}\times\vec{F})]\hat{e}_L$$

$$= \begin{vmatrix} e_x & e_y & e_z\\ r_x & r_y & r_z\\ F_x & F_y & F_z\end{vmatrix}$$

$$\vec{M}_L = (\hat{e}_L\cdot\vec{M}_o)\hat{e}_L$$

$$\boxed{\vec{M}_L = \underbrace{[\hat{e}_L\cdot(\vec{r}\times\vec{F})]}_{\text{scalar}}\hat{e}_L}$$

Special Cases:

$$\vec{M}_L = [\hat{e}_L\cdot(\vec{r}\times\vec{F})]\hat{e}_L$$

1. **F** is perpendicular to L.

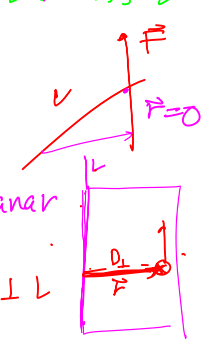
$$\boxed{M_L = F\cdot D_\perp}$$

2. **F** is parallel to L. ⇒ 0

3. **F** intersects L. ⇒ 0

Coplanar

$\vec{F}\perp L$



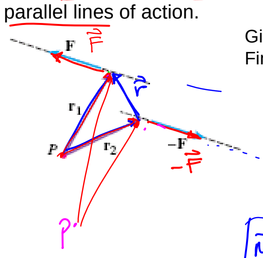
Couples

$\vec{r} = \vec{r}_1 - \vec{r}_2$

$\vec{r}_1 = \vec{r}_2 + \vec{r}$

Couple:

- Two forces of the same magnitude, opposite sense and parallel lines of action.

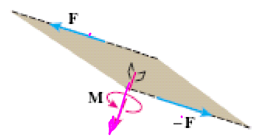


Given: **F** and **-F**

Find the moment of the couple about P.

$$\vec{M}_P = \vec{M}_{\vec{F}} + \vec{M}_{-\vec{F}}$$
$$= \vec{r}_1 \times \vec{F} + \vec{r}_2 \times (-\vec{F})$$
$$= \vec{r}_1 \times \vec{F} - \vec{r}_2 \times \vec{F}$$
$$= (\vec{r}_1 - \vec{r}_2) \times \vec{F}$$
$$\boxed{\vec{M}_P = \vec{r} \times \vec{F}}$$

Free Vector



The Couple Vector:

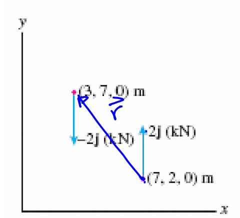
- -Perpendicular to plane of forces
- -A “free” vector.
- -Moment about any point is the same.

2-D

20N | 3m | 20N $\equiv$ 40N | 1.5m | 40N = 60 N-m

$M = 20(3)$ N-m = $40(1.5)$ N-m

Example 1. Determine the moment of the couple.



a) $M = (2\,kN)(4\,m)$

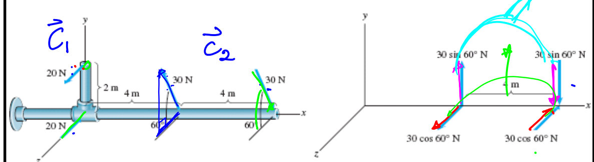
$= +8$ kN-m

b) $\vec{M} = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 5 & 0 \\ 0 & -2kN & 0 \end{vmatrix}$$

$= 0\hat{i} + 0\hat{j} + 8\hat{k}$ (kN)

Example 2. Find the sum of the moments exerted on the pipe by the two couples.



$\vec{C}_R = \vec{C}_1 + \vec{C}_2$

$\vec{C}_1 = +(20N)(2m)\hat{i}$

$\boxed{\vec{C}_1 = 40\,\hat{i}}$

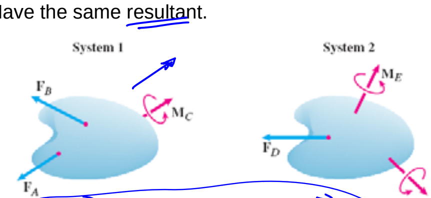
$\vec{C}_2 = +(30\cos 60°)4\;\hat{j} - (30\sin 60°)4\;\hat{k}$

$\boxed{\vec{C}_2 = 60\,\hat{j} - 120\sin 60°\,\hat{k}}$

$\vec{C}_R = [40\,\hat{i} + 60\,\hat{j} - 120\sin 60°\,\hat{k}]$ N-m

Equivalent Force Systems

- Have the same resultant.



3 eqns.

$$\sum \vec{F}_{sys1} \overset{(a)}{=} \sum \vec{F}_{sys2}$$

$$\sum (\vec{M}_P)_{sys1} = \sum (\vec{M}_P)_{sys2}$$

same point (P is arbitrary)

(eqns.

Example 3. The two force systems are equivalent. Determine the forces F_A and F_B and the couple M. Answers: 20N, 50N, -60 N-m

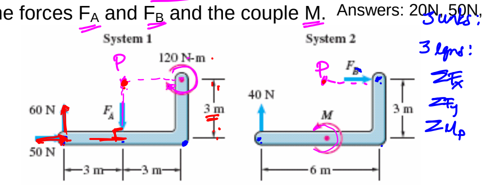
3 unks:

3 eqns:

$\sum F_x$

$\sum F_y$

$\sum M_P$



$(\sum F_x)_{sys1} = (\sum F_x)_{sys2}$

$50 = F_B \Rightarrow \boxed{F_B = 50N}$

$(\sum F_y)_{sys1} = (\sum F_y)_{sys2}$

$-F_A + 60 = 40 \Rightarrow \boxed{F_A = 20N}$

$\circlearrowleft +$ $(\sum M_P)_{sys1} = (\sum M_P)_{sys2}$

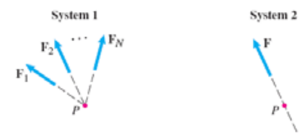
$+(50N)(3m) - (60N)(3m) - 120 = -40(3) + M$

$M =$

Equivalent Systems:

Concurrent Force System:

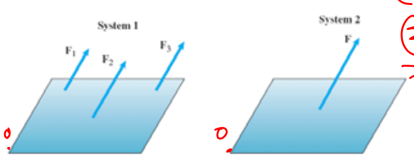
$(\Sigma F_x)_1 = (\Sigma F_x)_2$, y, z — 3 eqns.



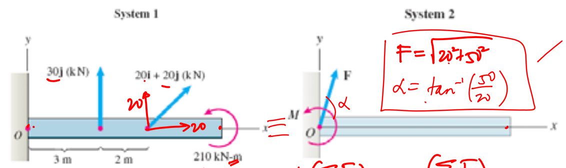
Parallel Force System:

$(\Sigma \vec{F})_1 = (\Sigma \vec{F})_2$

$(\Sigma M_O)_1 = (\Sigma M_O)_2$



Example 4. Represent the force system shown with:
a) A single force at the origin and a couple, b) A single ~~couple~~ FORCE



$\boxed{F = \sqrt{20^2 + 50^2}, \quad \alpha = \tan^{-1}\left(\frac{50}{20}\right)}$

$(\Sigma F_x)_{sys1} = (\Sigma F_x)_{sys2}$

$20 = F_x$

$(\Sigma F_y)_{sys1} = (\Sigma F_y)_{sys2}$

$30 + 20 = F_y$

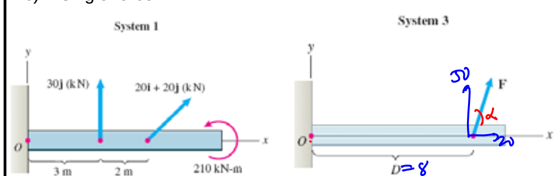
$50 = F_y$

$(\Sigma M_O)_{sys1} = (\Sigma M_O)_{sys2}$

$30(3) + 20(5) + 210 = M$

$\boxed{400 \text{ kN-m} = M}$

b) A single force



$20 = F_x$

$50 = F_y$

To find D:

$(\Sigma M_O)_{sys1} = (\Sigma M_O)_{sys2}$

$30(3) + 20(5) + 210 = 50\, D$

$\frac{400}{50} = D$

$8\text{m} = D$

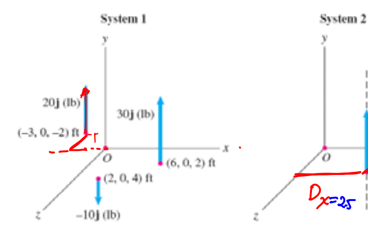
Example 5. Replace the given parallel force system by a single force.

$(\Sigma \vec{F})_{sys1} = (\Sigma \vec{F})_{sys2}$

$(\Sigma \vec{M})_1 = (\Sigma \vec{M})_2$

$\Sigma M_{x\text{-axis}}$

$\Sigma M_{z\text{-axis}}$



$(\Sigma F_y)_{sys1} = (\Sigma F_y)_{sys2}$

$\boxed{20 + 30 - 10 = F} \Rightarrow \boxed{F = 40\text{ lb}}$

$(\Sigma M_x)_{sys1} = (\Sigma M_x)_{sys2}$

$+20(2) - (30)2 + 10(4) = 40(D_z) \rightarrow D_z = 0.5\text{m}$

$(\Sigma M_z)_{sys1} = (\Sigma M_z)_{sys2}$

$-20(3) + 30(6) - 10(2) = 40 D_x \Rightarrow D_x = 2.5\text{m}$

The Wrench