Systems of Linear Equations: Solving by Determinants

A proof (where Cramer's Rule comes from.)

Outline: We'll solve a specific system of equations (with numbers) on the left and a general system (with letters) on the right, both by Linear Combination.

When we're done we should see a pattern in the general version (right side) which suggests a solution using determinants.

Example

Solve the system below:

$$\begin{aligned} 8x + 5y &= 23 \\ 3x + -2y &= 37 \end{aligned}$$

First we solve for x (by eliminating y):

$$\begin{aligned} (2)\quad 8x + 5y &= 23 \\ (5)\quad 3x + -2y &= 37 \end{aligned}$$

$$\begin{aligned} 16x + 10y &= 46 \\ +\ 15x + -10y &= 185 \\ 31x &= 231 \end{aligned}$$

dividing by 31:

$$\boxed{x = \frac{231}{31}}$$

Now we go back to the beginning and solve for y (instead of substituting this ugly fraction)

$$\begin{aligned} (3)\quad 8x + 5y &= 23 \\ (-8)\quad 3x + -2y &= 37 \end{aligned}$$

$$\begin{aligned} 24x + 15y &= 69 \\ +\ -24x + 16y &= -296 \\ 31y &= -227 \end{aligned}$$

Dividing by 31:

$$\boxed{y = \frac{-227}{31}}$$

The lines intersect at the point : $\left(\frac{231}{31}, \frac{-227}{31} \right)$

Solve the system below:

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

(where a, b, c, d, e, and f are just numbers)

First we solve for x (by eliminating y):

$$\begin{aligned} (e)\quad ax + by &= c \\ (-b)\quad dx + ey &= f \end{aligned}$$

$$\begin{aligned} aex + bey &= ce \\ +\ -bdx + -bey &= -bf \\ aex - bdx &= ce - bf \end{aligned}$$

Factoring out x:

$$x(ae - bd) = ce - bf$$

dividing by (ae – bd):

$$\boxed{x = \frac{ce - bf}{ae - bd}}$$

Now we go back to the beginning and solve for y (instead of substituting this ugly fraction)

$$\begin{aligned} (d)\quad ax + by &= c \\ (-a)\quad dx + ey &= f \end{aligned}$$

$$\begin{aligned} adx + bdy &= cd \\ +\ -adx + -aey &= -af \\ bdy - aey &= cd - af \end{aligned}$$

Factoring out y:

$$y(bd - ae) = cd - af$$

Dividing by (bd – ae):

$$y = \frac{cd - af}{bd - ae}$$

Notice the denominator is the reverse of the one for x (above) and it would look nice if they were the same, so we multiply top and bottom by –1:

$$y = \frac{-1}{-1} \bullet \frac{cd - af}{bd - ae} = \frac{af - cd}{ae - bd}$$

so

$$\boxed{y = \frac{af - cd}{ae - bd}}$$

The lines intersect at the point : $\left(\frac{ce - bf}{ae - bd}, \frac{af - cd}{ae - bd} \right)$

With the system
$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

You notice that the coordinates for the point of intersection are given by :

$$x = \frac{ce - bf}{ae - bd} \qquad \text{and} \qquad y = \frac{af - cd}{ae - bd}$$

Is there an easy way to find these values ? Yes! Using Determinants:

First, the denominator is the same for both fractions so

$$D = ae - bd = \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

Notice it's made up of the coefficients on the left side of the system :
$$\begin{aligned} \boxed{a}\,x + \boxed{b}\,y &= c \\ \boxed{d}\,x + \boxed{e}\,y &= f \end{aligned}$$

The Numerator of the y coordinate (N_x) is given by:

$$N_x = ce - bf = \begin{vmatrix} c & b \\ f & e \end{vmatrix}$$

Notice the x coefficients are replaced by the numbers on the right :
$$\begin{aligned} \boxed{c}\,x + \boxed{b}\,y &= c \\ \boxed{f}\,x + \boxed{e}\,y &= f \end{aligned}$$

The Numerator of the y coordinate (N_y) is given by:

$$N_y = af - cd = \begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

Notice the y coefficients are replaced by the numbers on the right :
$$\begin{aligned} \boxed{a}\,x + \boxed{c}\,y &= c \\ \boxed{d}\,x + \boxed{f}\,y &= f \end{aligned}$$

So the point of intersection is given by the coordinates:
$$\left(\frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}, \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \right)$$

Example:

Solve the system below using Determinants.
$$\begin{aligned} 6x - 7y &= 47 \\ 2x + 5y &= -21 \end{aligned}$$

$$D = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = \begin{vmatrix} 6 & -7 \\ 2 & 5 \end{vmatrix} = 30 - -14 = 44$$

$$\begin{aligned} \boxed{2x + 5y} &= -21 \\ \boxed{6x - 7y} &= 47 \end{aligned}$$

$$N_x = \begin{vmatrix} c & b \\ f & e \end{vmatrix} = \begin{vmatrix} 47 & -7 \\ -21 & 5 \end{vmatrix} = 235 - 147 = 88$$

$$\begin{aligned} \boxed{-21x + 5y} &= -21 \\ \boxed{47x - 7y} &= 47 \end{aligned}$$

replace x coefficients

$$N_y = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = \begin{vmatrix} 6 & 47 \\ 2 & -21 \end{vmatrix} = -126 - 94 = -220$$

$$\begin{aligned} \boxed{2x + -21y} &= -21 \\ \boxed{6x - 47y} &= 47 \end{aligned}$$

replace y coefficients

$$x = \frac{N_x}{D} = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} = \frac{88}{44} = 2 \qquad y = \frac{N_y}{D} = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} = \frac{-220}{44} = -5$$

So the point of intersection is at $(2, -5)$ or $S = \{(2, -5)\}$