Log Intro Name:________________

Show all relevant work!

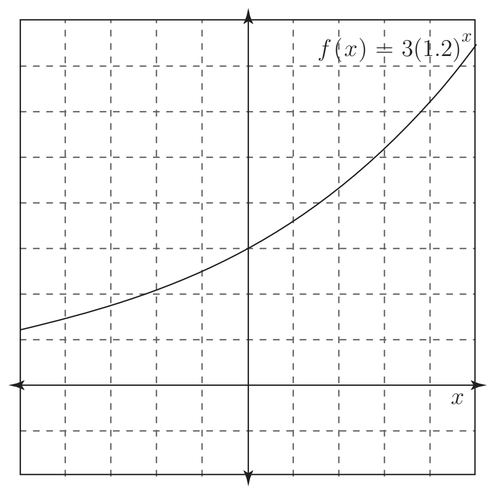
These problems are provided to help you develop a sense for solving exponential equations of the form $a \cdot b^x = k$. Follow the instructions carefully, and note that the first problems ask only for estimates.

0.1 Estimating Solutions to $a \cdot b^x = k$

1. Use the table below to help you *estimate* the solution to $7^x = 400$.

x	-1	0	1	2	3	4
$g(x) = 7^x$	$\frac{1}{7}$	1	7	49	343	2401

2. Use the graph to the right to help you *estimate* the solutions.



(a) $3(1.2)^x = 6$: ________

(b) $3(1.2)^x = 2$: ________

3. By guessing and checking, *estimate* the solutions to the equations below.

(a) $4^x = 20$

(b) $2^x = 0.1$

4. In 1910 the population of Bigtown was about 6,000 and in 1990 it was about 96,000. Assuming the growth of Bigtown is exponential, but without deriving a formula, *estimate* in which decade the population of Bigtown is over 150,000.

5. In 1910 the population of Metroville was about 500,000 and in 1990 it was about 31,000. Assuming the growth of Metroville is exponential, but without deriving a formula, *estimate* in which decade the population of Metroville is under 150,000.

0.2 Solving Exponential Equations With Common Bases

Let's begin with an example to see how we solve exponential equations where the bases are the same.

Solve: (a) $2^x = 8$ (b) $8^x = 32$

$$\begin{aligned} 2^x &= 2^3 \\ \longrightarrow x &= 3 \end{aligned}$$

$$\begin{aligned} (2^3)^x &= 2^5 \\ 2^{3x} &= 2^5 \\ \longrightarrow 3x &= 5 \\ x &= \frac{5}{3} \end{aligned}$$

The arrows indicate where we move from looking at the original equation to solving with the exponents.

$\boxed{\text{Moral: If the bases are the same, we can focus on the exponents.}}$

$\boxed{1.}$ Solve the following problems using the method described above.

(a) $9^x = 27$ (b) $32^x = 128$ (c) $125^x = \frac{1}{25}$

0.3 Logarithms

0.3.1 An investigation on your calculator

Press the LOG key on your calculator and type in 10, then press ENTER. Record the result: ____________

Press the LOG key on your calculator and type in 100, then press ENTER. Record the result: ____________

Press the LOG key on your calculator and type in 1000, then press ENTER. Record the result: ____________

Press the LOG key on your calculator and type in 10000, then press ENTER. Record the result: ____________

What does the LOG of a number seem to tell you?

Test your conjecture by trying out some more numbers. What do you expect the $\log(1,000,000)$ to be? $\log(0.1)$?

0.3.2 Solving Equations

What if we had to solve $17^x = 40$? Unlike the previous examples, the bases aren't obvious. If we *could* write both numbers as powers of the same base, we'd be in good shape.

0.3.3 History

Although their purposes were different from ours, the Arabs of the thirteenth century and Europeans of the seventeenth century, gave us the tools to do this. John Napier, while not the first to come up with the idea of writing numbers as powers, was the first to publish (around 1614 - he died in 1617). Unfortunately, he chose a really weird base - 0.9999999, and this made for some strange results. His friend, Henry Briggs, from Oxford, published a number of tables shortly after Napier's and these gave all the numbers as powers of various integers less than 1000 (accurate to 8 and later, 14, decimal places)! This is much like the trig tables produced from the same period. Today we use only a few of these bases because calculators and computers make it easy to change bases. In particular, the base 10 power table is stored in our calculators, much like the trig tables.
Because these tables consist entirely of powers of the same base, it's easier just to give the exponents with the base implied. Napier gave these numbers the name **logarithm** from the Greek, *logos*, meaning proportion, and *arismos*, or number. What powers have to do with proportions will be clearer later.

You've already seen a piece of the base 10 tables, once again, find $\log(17)$. What is this number? You probably already said something like, "It is the exponent we raise 10 to in order to get 17." Let's test it.
Type in $\boxed{\text{2nd}}$ $\boxed{\text{LOG}}$ to get 10^ and press $\boxed{\text{2nd}}$ $\boxed{\text{(-)}}$ to put in your last answer (the $\log(17)$).
We should have gotten 17 back and this tells us that $\log 17$ is the exponent I raise 10 to in order to get 17. Just like $\sin 30°$ is the number get when I divide the opposite side by the hypotenuse in a $30°$ right triangle.

0.4 Solving Exponential Equations Without Common Bases

This gives us all we need to solve $17^x = 40$:

<u>In Decimal Approximation</u>

$$\begin{aligned} 17^x &= 40 \\ (10^{1.230})^x &\approx 10^{1.602} \\ 10^{1.230x} &\approx 10^{1.602} \\ \longrightarrow 1.230x &\approx 1.602 \\ x &\approx 1.302 \end{aligned}$$

<u>In Symbols</u>

$$\begin{aligned} 17^x &= 40 \\ (10^{\log 17})^x &= 10^{\log 40} \\ 10^{(\log 17)x} &= 10^{\log 40} \\ \longrightarrow (\log 17)x &= \log 40 \\ x &= \frac{\log 40}{\log 17} \end{aligned}$$

$\boxed{1.}$ Use the method described above, *showing each step*, to solve these equations.

(a) $4^x = 20$ (b) $2^x = 0.1$ (c) $125^x = \frac{1}{25}$

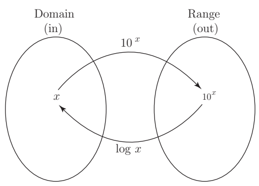
In introductory algebra classes we've learned to solve equations by "un-doing" operations in reverse of the order of operations. For example, to solve $3x - 5 = 17$, we first *add* 5 to both sides of the equation, and then *divide* by 3. Solving exponential equations is no different. We've seen that a logarithm is the inverse operation of exponentiation so we just add it to our list of operations that "un-do" each other.

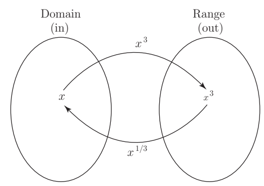
$\boxed{2.}$ Try o solve the following equations by reversing the order of operations.

(a) $3(1.2)^x = 6$ (b) $0.6^x + 7 = 15$ (c) $5(3)^x - 2 = 40$

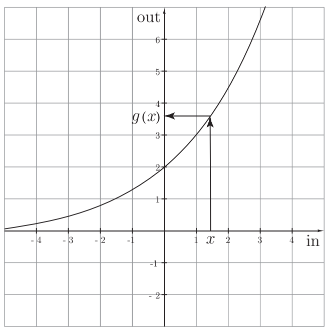
0.5 Log Properties

We've already noted that logarithms and exponential functions are inverses, much like $y = x^3$ and $y = \sqrt[3]{x}$.

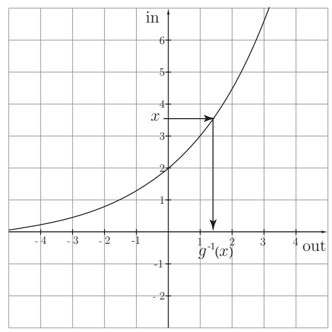




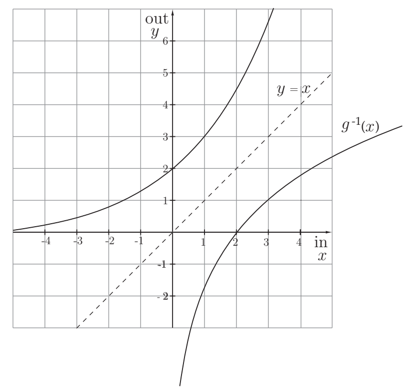
As we've seen with inverse functions in general, there are different views of the graphical representation:



$g(x) = 2(1.5)^x$



$g^{-1}(x)$
Axes reversed



$g(x), g^{-1}(x)$
Axes correct

The graph of $g(x) = 2(1.5)^x$ reminds us of the general shape of an exponential function. The center graph shows the idea of inverting $g(x) = 2(1.5^x$ by reversing the input and output axes. Switching the in and out values themselves will produce the graph of the inverse function in the usual orientation. This gives some sense of what the general shape of a logarithmic function looks like.

The general notation for logarithms is written: $y = \log_b x$ and read, y is the exponent I raise b to in order to produce x.

$$y = \log_b x$$

y is the exponent I raise b to in order to produce x

Note the equivalent form in exponential notation can be obtained by writing:

$$\log_b x = y \longrightarrow b^y = x$$

For example, $\log_3 81 = 4$ and $3^4 = 81$ are equivalent statements in the two different forms.

It's worth some time to practice going from logarithmic notation to exponential.

$\boxed{1.}$ Write these equations in exponential form and solve for x.

(a) $\log_2 16 = x$ (b) $\log_5 x = 2$ (c) $\log_x 64 = 3$