Logarithmic Scale

Orders of Magnitude

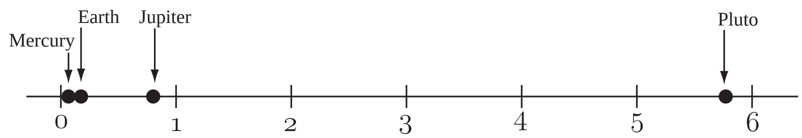
When we compare amounts, using absolute differences doesn't always convey the essential information. If Mike makes \$80,000 more than René the drama isn't nearly as dramatic when René is earning \$4,000,000 as when he's earning \$4,000. It is often more informative to speak in relative amounts. If Mike makes three times as much as René, then we get the picture regardless of the actual quantities involved. Comparison through relative amounts uses ratios, e.g. if Mike makes \$120,000 and René makes \$40,000, then Mike makes $\frac{120{,}000}{40{,}000} = 3$ times as much. When ratios of quantities are so big that they differ by powers of ten, then we call the relative size an *order of magnitude*. e.g. The distance from San Francisco to New York (3,000 miles) is two orders of magnitude greater than the distance from SF to Petaluma (30 miles). Mathematically this looks like $\frac{3000}{30} = 100 = 10^2 \longrightarrow 2$. Notice that the number of orders of magnitude is the exponent we raise 10 to in order to get the ratio. In symbols this means the order of magnitude for two quantities $= \log\left(\frac{Q_1}{Q_2}\right)$.

When numbers differ by large amounts it is impractical to try to fit them all on the same number line using absolute differences. However, using orders of magnitude allows us a way of representing their relative behavior.

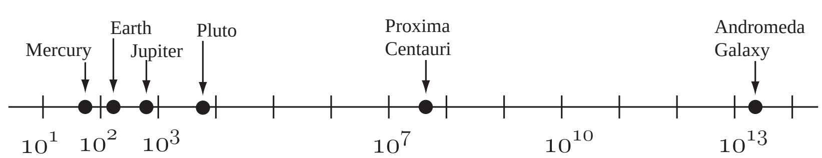
Example: The table below gives the distances for various planets and stars near our sun.

Object	Mercury	Earth	Jupiter	Pluto	Proxima Centauri	Andromeda Galaxy
Distance (in million km)	58	149	778	5900	4.1×10^7	2.4×10^{13}

A regular number line, marked in billions of kilometers, might squeeze the planets in:



But already the Earth and Mercury are on top of each other and the star, Proxima Centauri - the nearest star to the sun, would be 40,000 units away on the number line!
If we construct a number line that's graduated in powers of 10 instead, we get the following:



Note that this gives us a more complete picture but it requires some careful attention. The distances are relative rather than absolute. Proxima Centauri is near 10^7 while Earth is near 10^2, meaning that Proxima Centauri is five orders of magnitude, or 100,000 times as far from the sun as Earth (actually, it's $\frac{4.1\times 10^7}{149} \approx 280,000$ times as far). ■

The previous example gives some indication of why and how logarithms are used to model reality. Any quantities that span a large range of values are often modeled using logs. Three examples we will look at include the pH scale for acidity, decibels for sound, and the Richter scale for earthquakes.

pH

Acidity is measured using the pH scale which describes the number of moles of Hydronium ions (H_3O^+) per liter of water. The greater the concentration of H_3O^+, the greater the acidity. For pure water, the Hydrogen ion concentration is 1.0×10^{-7} mol(H_3O^+)/liter. For stomach acid it is about 1.0×10^{-2} and for Alka Seltzer™ it is about 1.0×10^{-10} mol(H_3O^+)/liter. Because the numbers of ions range over such large values, concentrations typically range between 10^{-14} moles/liter (very basic) and 1 mole per liter (very acidic), we focus on the exponents (note that the range is not limited to these values but most substances fall into this span).

The origin of the letters "p" and "H" is a point of some contention. While most agree that the "H" represents Hydrogen (or some equivalent), the "p" appears to have been an arbitrary choice made by the scientist who first used the term. We've come to interpret the "p" as either "power," or "potential."

> pH is defined as the opposite of the base 10 log of the hydrogen ion concentration in a substance.
>
> $$\text{pH} = -\log(\text{quantity mol}(H_3O^+))$$

Note that these concentrations represent extremely small numbers so all of the exponents are negative. The negative sign in the formula is included for convenience so that the numbers returned are positive.

Sound

The range of human hearing (like that of our sight) is enormous. The threshold of audible sound is about 1.0×10^{-16} watts/cm^2, while the threshold of pain for our hearing is about 10^{-4} watts/cm^2. That's a range of 12 orders of magnitude or equivalently, the loudest sound we can perceive is about 1,000,000,000,000 times as loud as the softest sound. The unit for measuring sound intensity comparing two sounds, I and I_0, is the decibel and by convention it is measured by the formula below.

> Decibels are units of relative intensity. The intensity of a sound is generally compared to the threshold of human hearing, $I_0 = 1.0 \times 10^{-16}$ watts/cm^2, and is given by the 10 times the log of their ratio.
>
> $$\text{Noise level in decibels} = 10 \cdot \log\left(\frac{I}{I_0}\right)$$

As in the case with pH, multiplying the results by ten is purely for convenience. This way the majority of sounds we examine in decibels will fall in to a range between 0 and 120 rather than 0 and 12.

A note about the **Richter Scale**.

Charles Richter introduced his scale in 1935 in order to differentiate between the magnitudes (and quantities) of earthquakes in southern California. The rough definition of the Richter scale is based in measuring the amplitude of the mark a quake registers on a seismograph and taking its log relative to some standard amplitude – much like the decibel scale. This means that the Richter scale is logarithmic and therefore every step in the scale corresponds to an increase in amplitude of a power of ten. So an 8 on the Richter scale has $10^4 = 10{,}000$ times the amplitude of a 4. Relating the amplitude of an earthquake and the energy released in causing it is a little less clear, however. The energy released in an earthquake is proportional to the square root of the cube of its amplitude and therefore each step in the Richter scale corresponds to an increase of $10^{3/2} \approx 31$ in energy. So the energy released in an earthquake registering 8 on the Richter scale is $(10^{3/2})^4 = 10^6 = 1,000,000$ times as great as the energy released in an earthquake registering a 4 on the Richter scale.

Seismologists generally reject the Richter scale and favor the Moment magnitude scale introduced in 1979:

> The Moment Magnitude Scale
>
> $$M = \frac{2}{3}\left(\log\left(\frac{M_0}{\text{dyn} \cdot \text{cm}} - 16.1\right)\right)$$
>
> Where M_0 is found using the amplitude of the quake, the area of the rupture and properties of the rock where it took place.

Since the Moment magnitude scale, like Richter, is a logarithmic function, the relative size of earthquakes measured by Moment scale behaves just like the Richter scale detailed above.

Problems

$\boxed{1.}$ The hydronium ion concentration of Drano® is $[H_3O^+] = 1.1 \times 10^{-12}$.
First estimate the pH of Drano® and then calculate the value using your calculator.

$\boxed{2.}$ Lemon Juice has a pH of 2. What is its hydronium concentration?

$\boxed{3.}$ Repeat #2 for Coke, with a pH of 4.5.

$\boxed{4.}$ A high concentration of hydronium ions corresponds to an acidic solution. As pH numbers increase, does the acidity increase or decrease? Explain.

$\boxed{5.}$ Using $I_0 = 1.0 \times 10^{-16}$ watts/cm^2 as the reference intensity, answer the following questions:

(a) How many decibels is a rock concert that registers 1.0×10^{-5} watts/cm^2?

(b) How many decibels is a normal conversation registering 1.0×10^{-11} watts/cm^2?

(c) How many times louder is the concert in (a) than the conversation in (b)?

$\boxed{6.}$ If a sound doubles in intensity, by how many units does its decibel rating increase.

$\boxed{7.}$ The intensity of sound and the power used to generate it are proportional.

(a) Assuming we perceive sound intensity logarithmically, how much louder (by what factor) will a 100 watt stereo seem than a 40 watt stereo?

(b) How much more powerful must a stereo be in order to be perceived as twice as loud as a 40 watt stereo?

$\boxed{8.}$ How much greater is the amplitude of an earthquake registering 7.3 on the Richter scale, than one registering 4.5? How much more energy is released?

$\boxed{9.}$ Solve.

a) $3 \cdot 5^x = 7 \cdot 11^x$

b) $\ln(3x + 2) = 2$

$\boxed{10.}$ The balance of an account is given by $P = 12,000(1.005)^{12t}$.

(a) Explain the meaning of each of the numbers in the formula.

(b) how long will it take before the balance reaches $20,000?

$\boxed{11.}$ The population of D-town is given by $P = 51(1.031)^t$ and the population of Big City is modeled by $P = 68(1.052)^t$, where P is in millions and t is in years.
After how many years will the populations of the two cities be the same?

$\boxed{12.}$ A super ball is dropped from a height of 24 feet. If the ball rebounds to 90% of its previous height each time it bounces, then find the following:

(a) Write a formula giving the height of the ball as a function of the number of bounces it has had.

(b) How high will the ball bounce on its 10^{th} bounce?

(c) How many bounces will the ball make before it rises to higher than 1 inch?