Chp. 5 Exam Review Answers

$\boxed{1.}$ a) Critical points occur where $f'(x) = 0$ or is undefined so at $x = -1$, $x = 1$, and $x = 3$
Local and global minima at $x = -1$, $x = 3$
Local maximum at $x = 1$
Global maxima at $x = -3$ and $x = 5$ (assuming these are endpoints).

b) Critical points occur where $f'(x) = 0$ or is undefined so at $x = -0.3$, $x = 0$, and $x = 0.3$ Notice the vertical tangent at $x = 0$ makes $f'(0)$ undefined so this is also a critical point.
Local minima at $x = \pm 3$ (assuming endpoints – otherwise no minima)
Global minimum at $x = -0.3$
Global maximum at $x = 0.3$

$\boxed{2.}$ a) Critical points occur where $f'(x) = 0$ or is undefined so at $x = -2$, $x = 1$, and $x = 4$
f' changes sign at $x = -2$ from $+ \to -$ so f has a local maximum at $x = -2$.
f' doesn't change sign at $x = 1$ so f has no extrema at $x = 1$.
f' changes sign at $x = 4$ from $- \to +$ so f has a local minimum at $x = 4$.

b) Critical points occur where $f'(x) = 0$ or is undefined so at $x = -1.8$ and $x = -1.15$
f' changes sign at $x = -1.8$ from $- \to +$ so f has a local minimum at $x = -1.8$.
f' changes sign at $x = -1.15$ from $+ \to -$ so f has a local maximum at $x = -1.15$.

$\boxed{3.}$ f has inflection points where the second derivative changes sign (and f is defined). f'' is the derivative of f' so we have inflection points where the slopes of the tangent lines to f' change sign.

a) This occurs at $x = -1$, $x = 1$, and $x = 3$

b) This occurs at $x = -1.5$, $x = -0.5$, and $x = 3$

$\boxed{4.}$ $g(x)$ increases fastest where $g'(x)$ (the rate at which $g(x)$ changes) is at a maximum. This occurs where the derivative of $g'(x)$ has a critical point and changes sign from $+ \to -$, that is, where $g''(x) = 0$ and changes from $+ \to -$. In this case That occurs at $x = -1.2$.
By similar logic, $g(x)$ decreases fastest at $x = -1.8$.
Concavity is related to the sign of the second derivative. $g''(x) < 0$ on $(-2, -1.8)$ and $(-1.2, 5)$ so this is where $(g(x)$ is concave down. Similarly, $g''(x) > 0$ on $(-1.8, -1.2)$ so this is where $(g(x)$ is concave up. Nothing can be told about the maximum or minimum values of g from this graph alone.

$\boxed{5.}$ We consider the endpoints and critical points of $h(x) = x^2 - \ln x$ on the interval $[.1, 2]$. The endpoints produce $h(0.1)) \approx 2.31$ and $h(2)) \approx 3.31$. The critical points occur where $h'(x) = 0$ or is undefined. $h'(x) = 2x - \frac{1}{x}$ which is undefined at $x = 0$ but we can ignore this since $0 \notin [0.1, 2]$. Solving $2x - \frac{1}{x} = 0$ we have $2x^2 = 1 \to x = \pm\sqrt{\frac{1}{2}}$ Since $\sqrt{\frac{1}{2}}$ is in the domain, this is the one critical point of $h(x)$. Evaluating gives $h\left(\sqrt{\frac{1}{2}}\right) \approx 0.85$ and by comparison with the end values we can conclude that there is a local maximum at $x = 0.1$, a global minimum at $x = \sqrt{\frac{1}{2}}$ and a global maximum at $x = 2$. By contrast, for the interval $(0, 2]$, there is no left endpoint and therefore no possibility of extrema at $x = 0$. The global minimum at $x = \sqrt{\frac{1}{2}}$ and the global maximum at $x = 2$ still remain.

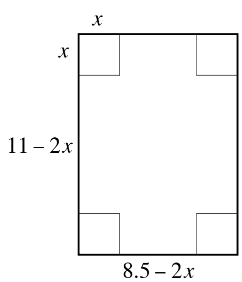
$\boxed{6.}$ In each case we consider critical points and long term behavior by taking limits at $\pm\infty$.
(a) $y' = 5 + 2x - 3x^2$ By QF (or factoring) there are critical points at $x = \frac{5}{3}$ and $x = -1$, Testing, y' changes sign $- \to +$ at $x = -1$ so we have a local minimum there and y' changes sign $+ \to -$ at $x = \frac{5}{3}$ so we have a local maximum there. $\lim_{x\to-\infty} y = \infty$ and $\lim_{x\to\infty} y = -\infty$ so the critical points give our only extrema and are only local extremes.

(b) $f'(x) = e^{-x} + x \cdot -e^{-x} = e^{-x}(1-x)$. Since $e^{-x} > 0$ for all x, the only critical point must occur when $x - 1 = 0$ so when $x = 1$. Testing shows f' changes sign from $+ \to -$ at $x = 1$ so we have a local maximum there. $\lim_{x\to\infty} \frac{x}{e^x} = 0$ and $\lim_{x\to-\infty} xe^{-x} = \lim_{x\to\infty} -xe^{x} = -\infty$. Therefore the maximum at $x = 1$ is global. (You can also see this from the graph)

(c) $g(x) = \frac{1}{x^2+1} = (x^2+1)^{-1}$ so $g'(x) = \frac{-2x}{(x^2+1)^2}$. The denominator is always positive so the only possible critical point occurs when numerator equals 0. This occurs when $x = 0$. Testing shows $g'(x)$ changes sign from $+ \to -$ so we have a local maximum at $x = 0$. $\lim_{x\to\infty} \frac{-2x}{(x^2+1)^2} = \lim_{x\to\infty} \frac{-2x}{x^4} = 0$. Similarly, $\lim_{x\to-\infty} \frac{-2x}{(x^2+1)^2} = \lim_{x\to-\infty} \frac{-2x}{x^4} = 0$. We conclude that since $g(0) = 1$, we have a global maximum there (since in the long run g goes to 0).

(d) $h'(x) = 3x^2 - \frac{2}{x}$ which is undefined at $x = 0$ but since $h(x)$ is undefined there also, 0 is not in the domain and can't be a critical point. $h'(x) = 0$ results in $x = \sqrt[3]{\frac{2}{3}}$. Testing shows $h'(x)$ changes sign from $- \to +$ so we have a local minimum at $x = \sqrt[3]{\frac{2}{3}}$. $\lim_{x\to\infty} h(x) = \infty$ and $\lim_{x\to-\infty} h(x) = -\infty$ so the minimum at $x = \sqrt[3]{\frac{2}{3}}$ is local.

$\boxed{7.}$ Label the rectangle as shown, with x representing the length of the side of the square that is to be removed. We want to maximize the volume of the box made when the sides are folded up. That is, we want to maximize $V = \ell \cdot h \cdot w$. The dimensions are constrained by the choice of x and we can see that $\ell = 11 - 2x$, $w = 8.5 - 2x$, and $h = x$ so we can write the formula to maximize in terms of the variable x as $V = x(11 - 2x)(8.5 - 2x) = 4x^3 - 39x^2 + 93.5x$.
The domain is controlled by reasonable choices for x. x shouldn't be negative so we have $x \geq 0$, and once x exceeds half the width, we no longer have 4 squares to cut out so $x \leq 4.25$. The domain is $[0, 4.25]$. Testing endpoints gives $V(0) = 0$ and $V(4.25) = 0$.
We now consider critical points. $V'(x) = 12x^2 - 78x + 93.5$ and solving $12x^2 - 78x + 93.5 = 0$ gives $x = 1.59$ and $x = 4.91$. We can ignore $x = 4.91 \notin [0, 4.25]$ and evaluating $V(1.59) = 66.15$ so the maximum volume occurs when we cut out 1.59 inch squares.



$\boxed{8.}$ We know $\frac{\mathrm{d}r}{\mathrm{d}t} = 2$.
We want $\frac{\mathrm{d}A}{\mathrm{d}t}$.
Relating A and r comes from $A = \pi r^2$ and from this if we take $\frac{\mathrm{d}}{\mathrm{d}t}$ of this equation we get $\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \frac{\mathrm{d}r}{\mathrm{d}t}$.
Evaluating this for $r = 6$ we have $\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi(6)(2) = 24\pi$ ft^2/sec.

$\boxed{9.}$ From half-life problems in the book we have already seen that the quantity of Carbon–14 at time t is given by $Q = Q_0 e^{-0.000121t}$.
From this it follows that the (continuous) relative decay rate is 0.0121%. The absolute rate of decay of 10 grams after 500 years is given by the derivative of Q: $Q' = -0.000121 \cdot 10e^{-0.000121(500)} \approx -0.0011$ grams/year.

$\boxed{10.}$ The points (12, 70) and (9, 98) yield the demand function $q = -\frac{28}{3}p + 182$. From this we have $R = pq = -\frac{28}{3}p^2 + 182p$. Then for the domain $p \geq 0$ we have only the critical point, $-\frac{56}{3}p + 182 = 0 \rightarrow p = 9.75$ to consider. Note that $R'(p)$ changes sign $+ \rightarrow -$ around $p = 9.75$ so there is a maximum in revenue when we sell pizzas for \$9.75 each. Also notice that we get the same result using elasticity and setting $E = 1 = -\frac{\mathrm{d}q}{\mathrm{d}p} \cdot \frac{p}{q}$.