Exponential Functions

Exponential functions have the form $y = a \cdot b^x$ where we assume $b > 0$ (you might test some examples with $b < 0$ just to see why we make this restriction). There is also a more general representation, $y = a \cdot b^x + c$ (what does the c do?). Let's begin our review with some graphical representations.

Graphical

$\boxed{1.}$ Sketch graphs for the equation $y = a \cdot b^x$ based on the parameters given below.

(i) $a < 0$, $0 < b < 1$ (ii) $a > 0$, $0 < b < 1$ (iii) $a < 0$, $b > 1$ (iii) $a > 0$, $b > 1$

$\boxed{2.}$ The functions below are all of the form $y = a \cdot b^x + c$. For each graph find values for a, b, and c that produce a graph like the one shown.

(a) (b) (c) (d)

We've also seen how the data tables for exponential, linear, and quadratic functions behave.

Numerical

$\boxed{3.}$ Suppose $f(x)$ is quadratic, $g(x)$ is exponential of the form $y = a \cdot b^x$, and $h(x)$ is linear.
Complete each of the tables below to match these functions.

x	1	2	3	4	5
$f(x)$	14				

x	1	2	3	4	5
$g(x)$	14				

x	1	2	3	4	5
$h(x)$	14				

It's worth reviewing your notes to see examples working from given data to recognize the function type and find an appropriate formula. Consider the situation described below:

Applied:

$\boxed{4.}$ The issue of global warming is becoming increasingly difficult to ignore. Data from monitoring the decay of the Greenland Ice Sheet show that in 1996 the ice sheet was melting at a rate of 96 km^3/year while in 2005 ice was disappearing at 220 km^3/year.

(a) Suppose the rate at which ice is melting in the Greenland Ice Sheet is a linear function and write its equation in terms of years since 1996.

(b) Repeat part (a) but this time assume the function is exponential.

(c) The total ice contained in the Greenland Ice Sheet is estimated to be about 2.6×10^6 km^3.
Discuss the significance of using the linear versus the exponential model to address this fact.

Interest

We often see the use of exponential models for financial situations. Recall why this makes sense by looking at the model for simple interest compounded over t years:

$\boxed{5.}$ If the annual percentage rate (APR) for an account is 5%, what is the balance on a \$1000 investment at the end of one year? The solution is equivalent to multiplying $1000(1 + 0.05)$, the one returns the original balance and the 0.05 adds the interest. Therefore the balance is given by multiplying $1000(1.05) = \$1050$. For the following year the balance would be given by $1050 \times (1.05)$ which is the same as $1000(1.05) \times (1.05)$ or $1000(1.05)^2$. The table below shows the balances for the subsequent years.

t (years)	Balance (\$)
0	\$1000.00
1	$1000(1.05) = \$1050.00$
2	$1000(1.05)(1.05) = \$1102.50$
3	$1000(1.05)(1.05)(1.05) = \$1157.63$
4	$1000(1.05)^4 = \$1215.51$
$\vdots$	$\vdots$
t	________________

The general term in the table above looks like $1000(1.05)^t$ and in general for principal, P_0 (the starting amount), an annual interest rate, r, compounded over t years, we get $P = P_0(1+r)^t$.

Since interest is generally compounded more often than once a year we break the year up into pieces. If we have an annual interest rate of r, then the interest for one month would be $\frac{1}{12}r$. If we compound the interest monthly, then we will compute the interest 12 times in one year. If we keep r as the annual interest rate, and t in years, then compounding monthly becomes $P = P_0(1+\frac{r}{12})^{12t}$. Check to see that the balance of an account with \$1000 invested at 5% compounded monthly for 1 year is \$1051.16. Or \$1.16 more than if we had simple interest.

This gives us the general formula for compound interest, or

$$P = P_0\left(1+\frac{r}{n}\right)^{nt} \tag{1}$$

Where t and r both use the same unit of time and n is the number of compoundings over that unit.

We've pursued the idea that if a little compounding is good, then maybe a lot is better. The result is better indeed, but not infinitely better as we might imagine. In fact, by letting $n \to \infty$ in the expression $\left(1+\frac{1}{n}\right)^n$, we see that we get an irrational number approximated by 2.71828 . . . which we call e.

This gives us our second equation for compounding, but where (1) is for discrete compoundings, (2) below is continuously compounded.

$$P = P_0 e^{rt} \tag{2}$$

$\boxed{6.}$ If you invest \$1200 at 5% compounded quarterly, how much will your investment be worth in 3 years?

$\boxed{7.}$ If you take out a loan for \$5000 at 8% compounded monthly, what is the equivalent APR for one year? (Ignore monthly payments).

Invariably, exponential functions lead to questions of inverse relationships like,

$\boxed{8.}$ If you invest \$1200 at 5% compounded quarterly, how long will it be before your investment is worth \$5000?

This brings us to logarithms and the notes for them are provided with section 1.4. However, let's assume that this is *review* and for the sake of continuing the narrative, we will proceed with more applications.

$\boxed{9.}$ We saw in #1 how the parameters a and b affect the graph of $y = a \cdot b^x$. Similarly determine conditions for the effects of the parameters a and k in $y = a \cdot e^{kx}$.

$\boxed{10.}$ If a population grows from 2,000 in 1990 to 7,000 in 2005, write an equation for the population as a function of time since 1990 and determine both the continuous and annual growth rates for this population.

$\boxed{11.}$ The population of bacteria in a dish doubles in 5 hours. Assuming their growth is exponential, how long will it take the population to triple?

$\boxed{12.}$ In the early 1960's, radioactive strontium-90 was released during atmospheric testing of nuclear weapons and got into the bones of people alive at the time. If the half-life of strontium-90 is 29 years, what fraction of the strontium-90 absorbed in 1960 remained in people's bones in 2000?