0.1 Limits

The technical definition of a limit is based on the idea that if we have correctly identified the limiting quantity, then we should be able to get the function output as close to it as we want by making the input values close enough to the limit index.

0.1 Limits

The technical definition of a limit is based on the idea that if we have correctly identified the limiting quantity, then we should be able to get the function output as close to it as we want by making the input values close enough to the limit index.

The naive interpretation means that if $\lim_{x \to 5} x^2 = 25$, then we should be able to find a value of x close enough to 5 so that we can be as close as we want to 25.

If, for example, we wanted to be within 0.01 of 25 (think of tolerances), then we would solve $|x^2 - 25| < 0.01$ or equivalently, $-0.01 < x^2 - 25 < 0.01$.

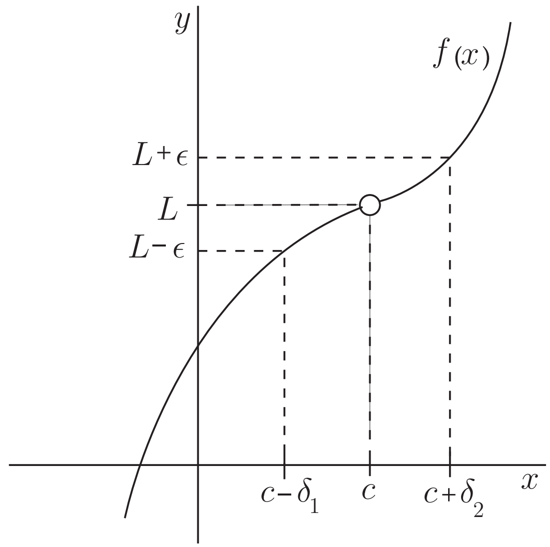
Solving gives us $\sqrt{24.99} < x < \sqrt{25.01}$
or about $4.9989999 < x < 5.0009999$. Subtracting 5 gives us $-0.0010001 < x - 5 < 0.0009999$ and since the smaller side of the interval (0.0009999) guarantees a safe solution, we know x^2 will always be within 0.01 of 25 if x is within 0.0009999 of 5.

0.1 Limits

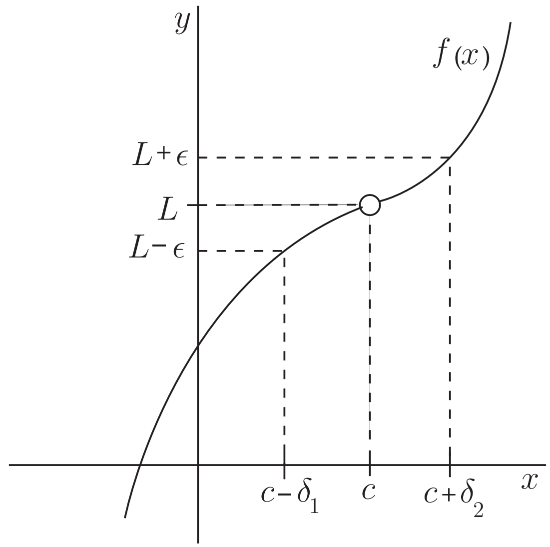
The technical definition of a limit is based on the idea that if we have correctly identified the limiting quantity, then we should be able to get the function output as close to it as we want by making the input values close enough to the limit index.

> The naive interpretation means that if $\lim_{x \to 5} x^2 = 25$, then we should be able to find a value of x close enough to 5 so that we can be as close as we want to 25.
>
> If, for example, we wanted to be within 0.01 of 25 (think of tolerances), then we would solve $|x^2 - 25| < 0.01$ or equivalently, $-0.01 < x^2 - 25 < 0.01$.
>
> Solving gives us $\sqrt{24.99} < x < \sqrt{25.01}$
> or about $4.9989999 < x < 5.0009999$. Subtracting 5 gives us $-0.0010001 < x - 5 < 0.0009999$ and since the smaller side of the interval (0.0009999) guarantees a safe solution, we know x^2 will always be within 0.01 of 25 if x is within 0.0009999 of 5.

While the example above gives some idea of the process for proving a limit exists, it is only practical to use numerical methods when applying the idea of tolerances or a similar application. The actual definition of a limit was designed to create a sound basis for the theory of calculus. The proper definition follows:

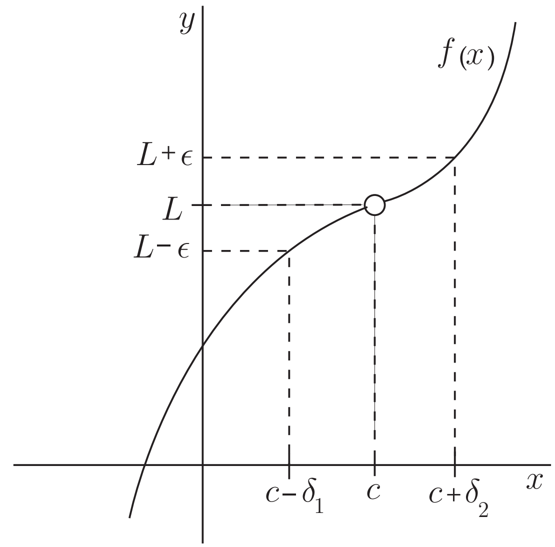


Definition: The limit $\lim_{x \to c} f(x)$ is defined to be the number L (if it exists) such that for any $\epsilon > 0$ we choose, there is a $\delta > 0$ where $|x - c| < \delta$ (but $x \neq c$) guarantees that $|f(x) - L| < \epsilon$.



Definition: The limit $\lim_{x \to c} f(x)$ is defined to be the number L (if it exists) such that for any $\epsilon > 0$ we choose, there is a $\delta > 0$ where $|x - c| < \delta$ (but $x \neq c$) guarantees that $|f(x) - L| < \epsilon$.

The proof of $\lim_{x \to 5} x^2 = 25$ would therefore be directed to finding a general rule (formula) that would assure a suitable δ for any choice of ϵ. This is sometimes described as winning the $\epsilon - \delta$ game. (Every time you declare a bounding value ϵ, I have to be able to respond with a suitable δ to satisfy the definition.



Definition: The limit $\lim_{x \to c} f(x)$ is defined to be the number L (if it exists) such that for any $\epsilon > 0$ we choose, there is a $\delta > 0$ where $|x - c| < \delta$ (but $x \neq c$) guarantees that $|f(x) - L| < \epsilon$.

The proof of $\lim_{x \to 5} x^2 = 25$ would therefore be directed to finding a general rule (formula) that would assure a suitable δ for any choice of ϵ. This is sometimes described as winning the $\epsilon - \delta$ game. (Every time you declare a bounding value ϵ, I have to be able to respond with a suitable δ to satisfy the definition.

We begin by stating our desired result: $|x^2 - 25| < \epsilon$. Then we have $|x-5||x+5| < \epsilon$ or $|x - 5| < \frac{\epsilon}{|x+5|}$. Since we want $\delta < |x - 5|$, this means $\delta < \frac{\epsilon}{|x+5|}$. The only problem is that this answer depends on x and we want a guarantee that for *any* x if δ is small enough then $|x^2 - 25| < \epsilon$. Note that we want x close to 5. If we agree that x should be at least within 1 unit of 5, then we have $|x - 5| < 1$ and since
$|x| - 5 < |x - 5| \longrightarrow |x| - 5 < 1 \longrightarrow |x| < 6$.
Since $|x + 5| \leq |x| + 5$ it follows from above that $|x + 5| < 11$ (for x within 1 unit of 5). Then we have $|x - 5| < \frac{\epsilon}{11}$. So if we set $\delta = \frac{\epsilon}{11}$, then for *any* $\epsilon > 0$ (but not too big) we have a $\delta = \frac{\epsilon}{11}$ that will guarantee that $|x^2 - 25| < \epsilon$. Note than in our numerical example, this would give us $\delta = \frac{0.1}{11} \approx 0.000909$ which is just within the boundaries we set for δ.

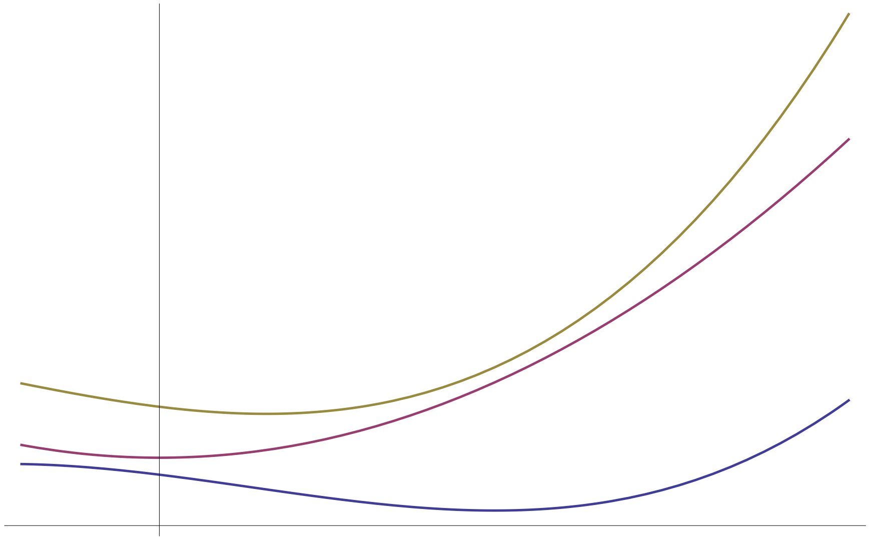
Consider the limit below and try to repeat this example, first numerically and then symbolically.

Exercise: $\lim_{x \to 3} 2x = 6$

Limit of a Sum

Theorem:

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} (f(x) + g(x)) = L + M$



Theorem:

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}(f(x) + g(x)) = L + M$

Proof:

Since $\lim_{x\to a} f(x) = L$ we know there is a δ_1 such that $|f(x) - L| < \epsilon/2$ when $|x - a| < \delta_1$. Similarly, since $\lim_{x\to a} g(x) = M$ we know there is a δ_2 such that $|g(x) - M| < \epsilon/2$ when $|x - a| < \delta_2$. Let $\delta = \text{minimum}(\delta_1, \delta_2)$. Then for $|x - a| < \delta$ we have

$$|(f(x) + g(x)) - (L + M)| = |f(x) - L + g(x) - M| \tag{1}$$

$$\leq |f(x) - L| + |g(x) - M| \tag{2}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{3}$$

Example:

Show $\lim_{x \to 0} \frac{x}{|x|}$ does not exist (there is no limit).

Example:

Show $\lim_{x \to 0} \frac{1}{x}$ does not exist (there is no limit).

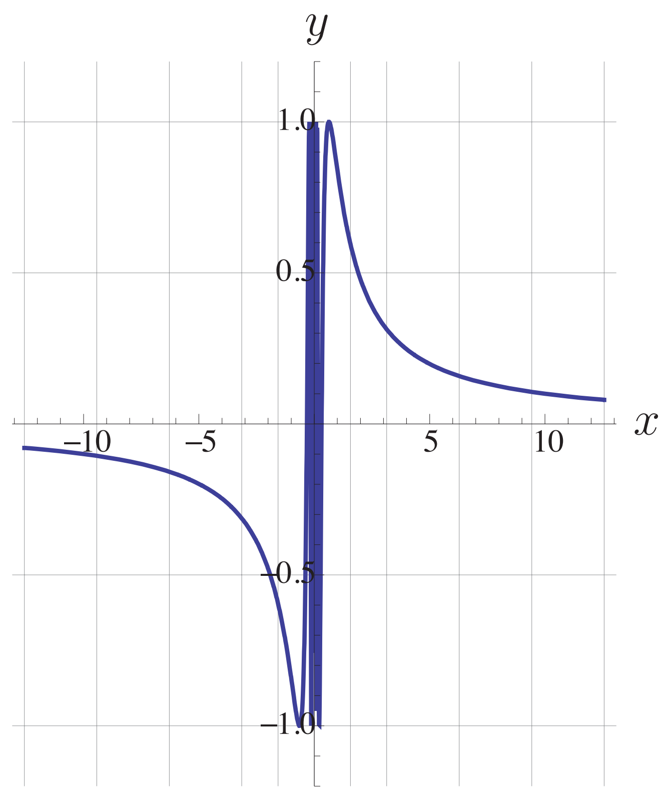
Example:

Show $\lim_{x \to 0} \frac{1}{x^2}$ does not exist (infinite limit).

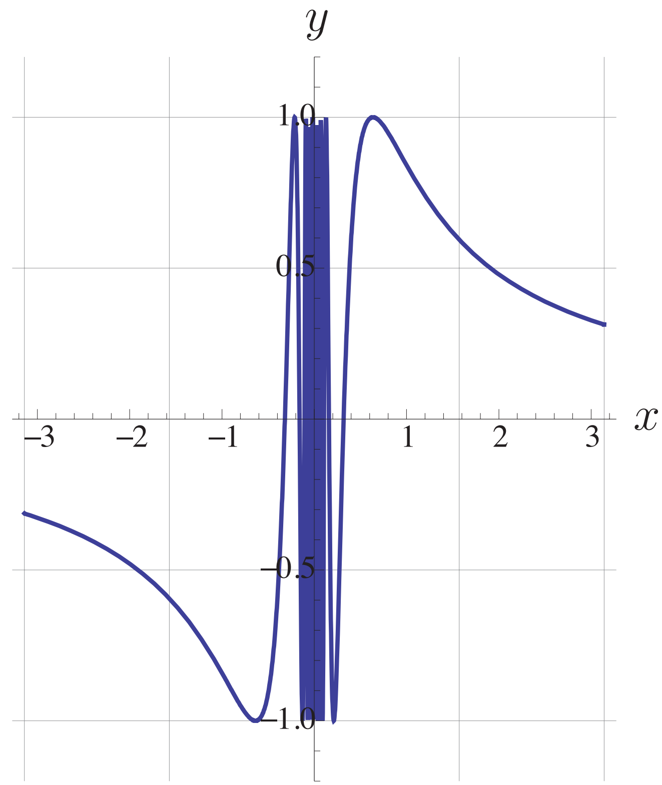
Example:

Consider

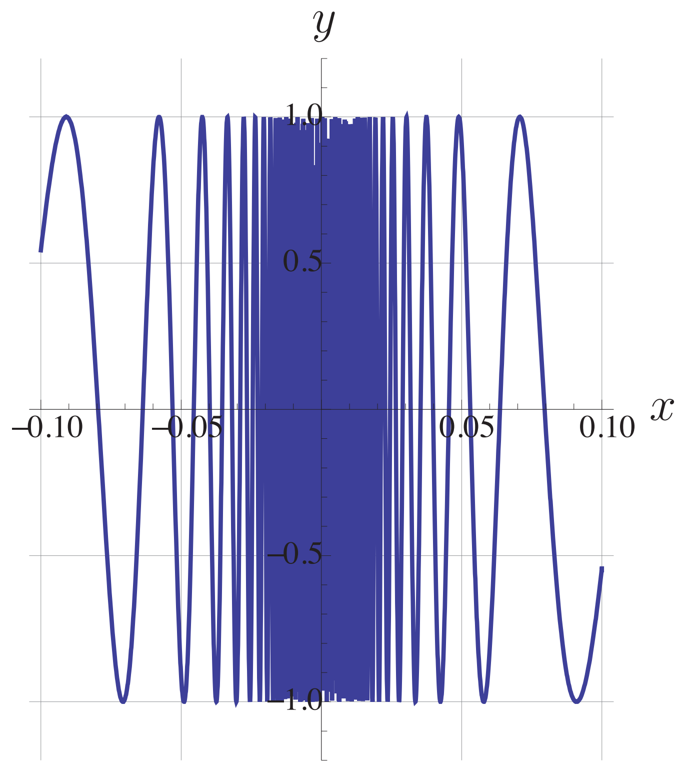
$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$



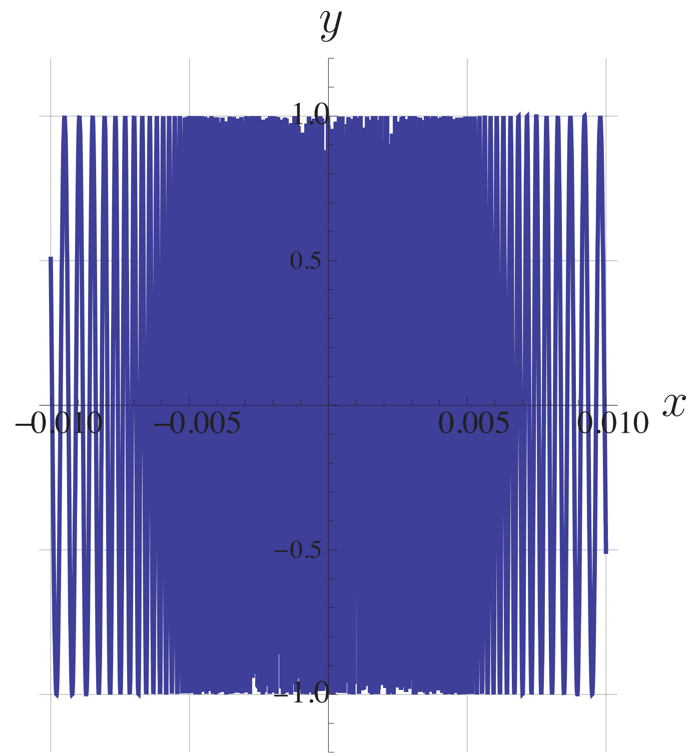
$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$



$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$



$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$



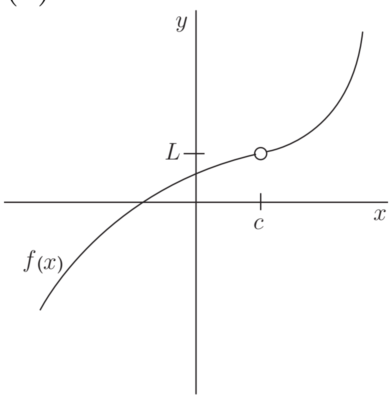
0.2 Continuity

The definition of a continuous function is given below.

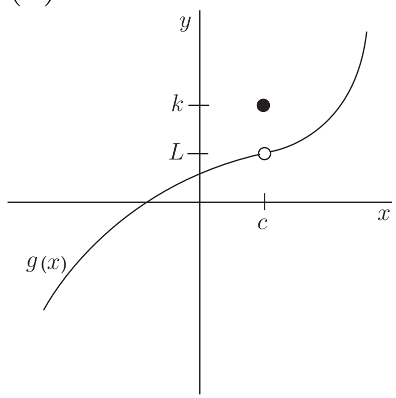
Definition: The function f is continuous at $x = c$ if f is defined at $x = c$ and if $\lim_{x \to c} f(x) = f(c)$.

The diagrams below show a variety of discontinuities. In each case, explain which condition of the definition above the function pictured is violating.

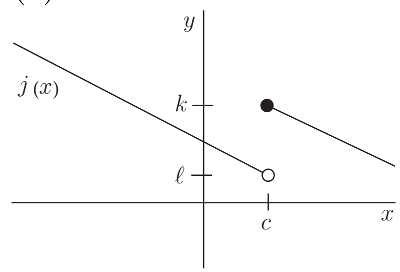
(a)



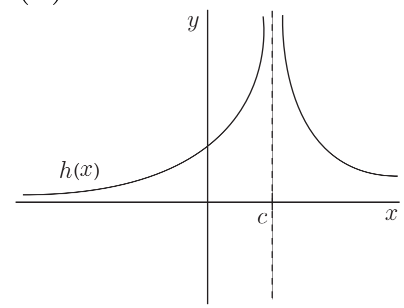
(b)



(c)



(d)



0.2.1 Intermediate Value Theorem

Suppose f is continuous on the closed interval $[a, b]$. If $f(a) < k < f(b)$ then there is at least one number c in $[a, b]$ such that $f(c) = k$.

Proofs of the properties of continuity are relatively straight forward. For example, if $f(x)$ and $g(x)$ are continuous, prove that $f(x) \cdot g(x)$ is continuous.

proof: Let $h(x) = f(x) \cdot g(x)$. We want to show that $\lim_{x \to c} h(x) = h(c)$. From the properties of limits, we know that $\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$ so it follows that $\lim_{x \to c} h(x) = \lim_{x \to c} f(x) \lim_{x \to c} g(x) = f(c)g(c)$ (since f and g are continuous) $= h(c)$.

Exercise: Prove that if $f(x)$ and $g(x)$ are continuous, then $f(x) + g(x)$ is continuous.