

0.1 Limit Properties

The list below includes common properties of limits.

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| (1) $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$ | (4) $\lim_{x \rightarrow a} kf(x) = k \lim_{x \rightarrow a} f(x), k \in \mathbb{R}$ |
| (2) $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = (\lim_{x \rightarrow a} f(x))(\lim_{x \rightarrow a} g(x))$ | (5) $\lim_{x \rightarrow a} k = k$ |
| (3) $\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ | (6) $\lim_{x \rightarrow a} x^n = a^n, n \in \mathbb{N}$ |

0.1.1 Limits at Infinity

- (1) $\lim_{x \rightarrow \infty} x = \infty, \quad \lim_{x \rightarrow \infty} e^x = \infty$
- (2) $\lim_{x \rightarrow \infty} \frac{1}{x} = 0, \quad \lim_{x \rightarrow \infty} e^{-x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$
- (3) $\lim_{x \rightarrow a} x^n = a^n, n \in \mathbb{N}$

0.1.2 Infinite Limits

- (1) $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty, \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$ So $\lim_{x \rightarrow 0} \frac{1}{x}$ is undefined.

0.2 Limit at a Finite Value

Example: Find $\lim_{x \rightarrow 3} \frac{x^3 - x^2 + x - 21}{x^2 - x - 6}$

First check to see if direct substitution produces the indeterminate form $\frac{0}{0}$.
 (If not, the form $\frac{n}{0}, (n \neq 0)$ goes to $\pm\infty$ while the form $\frac{0}{n}, (n \neq 0)$ goes to 0.)

Since $\lim_{x \rightarrow 3} \frac{x^3 - x^2 + x - 21}{x^2 - x - 6}$ has the form $\frac{0}{0}$, we proceed by trying to factor and simplify:

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^3 - x^2 + x - 21}{x^2 - x - 6} &= \lim_{x \rightarrow 3} \frac{x^3 - x^2 + x - 21}{(x-3)(x+2)} \rightarrow \text{(factor top by long division)} \rightarrow \lim_{x \rightarrow 3} \frac{(x-3)(x^2 + 2x + 7)}{(x-3)(x+2)} \\ \lim_{x \rightarrow 3} \frac{\cancel{(x-3)}(x^2 + 2x + 7)}{\cancel{(x-3)}(x+2)} &= \lim_{x \rightarrow 3} \frac{x^2 + 2x + 7}{x+2} = \frac{(3)^2 + 2(3) + 7}{(3) + 2} = \frac{22}{5} \quad \square \end{aligned}$$

0.3 Limit at Infinity

Example: Find $\lim_{x \rightarrow \infty} \frac{x^3 - 2x^2 + 3x - 1}{3x^3 + x}$

Begin by dividing each term by the largest power among the terms (x^3):

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^3 - 2x^2 + 3x - 1}{3x^3 + x} &= \lim_{x \rightarrow \infty} \frac{\frac{x^3}{x^3} - \frac{2x^2}{x^3} + \frac{3x}{x^3} - \frac{1}{x^3}}{\frac{3x^3}{x^3} + \frac{x}{x^3}} = \frac{\lim_{x \rightarrow \infty} \frac{x^3}{x^3} - \lim_{x \rightarrow \infty} \frac{2x^2}{x^3} + \lim_{x \rightarrow \infty} \frac{3x}{x^3} - \lim_{x \rightarrow \infty} \frac{1}{x^3}}{\lim_{x \rightarrow \infty} \frac{3x^3}{x^3} + \lim_{x \rightarrow \infty} \frac{x}{x^3}} \\ &= \frac{1 - 0 + 0 - 0}{3 + 0} = \frac{1}{3} \quad \square \end{aligned}$$

Equivalently, this amounts to a shortcut. Since x^3 grows faster than x^2 or x , for big enough values of x ,
 As $x \rightarrow \infty, \frac{x^3 - 2x^2 + 3x - 1}{3x^3 + x} \approx \frac{x^3}{3x^3}$. So $\lim_{x \rightarrow \infty} \frac{x^3 - 2x^2 + 3x - 1}{3x^3 + x} = \lim_{x \rightarrow \infty} \frac{x^3}{3x^3} = \lim_{x \rightarrow \infty} \frac{\cancel{x^3}}{3\cancel{x^3}} = \frac{1}{3} \quad \square$

0.4 Horizontal Asymptotes

Horizontal asymptotes are limiting values of a function over a long period (positive or negative). Consequently, they're found by taking the infinite limits $y = \lim_{x \rightarrow \infty} f(x)$ and $y = \lim_{x \rightarrow -\infty} f(x)$.

Example: The horizontal asymptotes of $f(x) = \frac{2e^x + 3e^{-x}}{3e^x + 2e^{-x}}$ are given by,

$$y = \lim_{x \rightarrow \infty} \frac{2e^x + 3e^{-x}}{3e^x + 2e^{-x}} \rightarrow \frac{2e^x + 3\cancel{e^{-x}}^0}{3e^x + 2\cancel{e^{-x}}^0} = \lim_{x \rightarrow \infty} \frac{2\cancel{e^x}}{3\cancel{e^x}} = \frac{2}{3} \quad \text{and} \quad y = \lim_{x \rightarrow -\infty} \frac{2e^x + 3e^{-x}}{3e^x + 2e^{-x}} = \lim_{x \rightarrow -\infty} \frac{2e^{-x} + 3e^x}{3e^{-x} + 2e^x} = \frac{3}{2} \quad \square$$