3.1 Notes

1. Some elementary derivatives.

\( \frac{d}{dx} k \): The derivative of a constant.

Some arguments for \( \frac{d}{dx} k = 0 \):

- **Graphically:** The derivative gives the slope of the function at \( x \).
  For a constant function, the slope is 0 so \( \frac{d}{dx} k = 0 \).

- **Numerically:**

  \[
  \begin{array}{cccccc}
  x & -3 & 0 & 3 & 6 & 9 \\
  f(x) & k & k & k & k & k \\
  \hline
  & 0 & 0 & 0 & 0 & \\
  \end{array}
  \]

- **Algebraically (from definition):**

  \[
  \frac{d}{dx} k = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{k - k}{h} = \lim_{h \to 0} 0 = 0
  \]

\( \frac{d}{dx} (mx + b) \): The derivative of a linear function.

- **Graphically:** The derivative gives the slope of the function at \( x \).
  For a linear function, the slope is \( m \) so \( \frac{d}{dx} (mx + b) = m \).

- **Numerically:**

  \[
  \begin{array}{ccccccc}
  x & -1 & 0 & 1 & 2 & 3 \\
  f(x) & -m + b & 0 + b & m + b & 2m + b & 3m + b \\
  \hline
  y & m & m & m & m & \\
  \end{array}
  \]

- **Algebraically (from definition):**

  \[
  \frac{d}{dx} (mx + b) = \lim_{h \to 0} \frac{m(x + h) + b - (mx + b)}{h} = \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m
  \]

2. Some derivative properties

\( \frac{d}{dx} kf(x) \): The derivative of a constant times a function.

- **Numerical intuition:** Find the slopes between each pair of points to approximate the derivative of \( f(x) \):

  \[
  \begin{array}{cccc}
  x & -3 & 0 & 3 & 6 & 9 \\
  f(x) & 10 & 8 & 1 & 5 & 11 \\
  \end{array}
  \]

  Now repeat for the function \( y = 5f(x) \):

  \[
  \begin{array}{cccc}
  x & -3 & 0 & 3 & 6 & 9 \\
  5f(x) & 50 & 40 & 15 & 25 & 55 \\
  \end{array}
  \]

  How are the slopes of the two tables related?

The proof follows similar logic but uses limit properties and the definition of a derivative function which may make it seem more abstract.

Remember the definition of the derivative function is \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)

**Proof:**

\[
\frac{d}{dx} kf(x) = \lim_{h \to 0} \frac{kf(x + h) - kf(x)}{h} = \lim_{h \to 0} \frac{k[f(x + h) - f(x)]}{h} = k \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = kf'(x)
\]
\[
\frac{d}{dx}(f(x) + g(x)):\text{ The derivative of the sum of functions.}
\]

Numerical intuition: Find the slopes between each pair of points to approximate the derivatives of \(f(x)\) and \(g(x)\):

| \(x\) | \(-3\) | \(0\) | \(3\) | \(6\) | \(9\) |
| \(f(x)\) | 10 | 8 | 1 | 5 | 11 |
| \(g(x)\) | -2 | 1 | 6 | 15 | 27 |

Now repeat for the function \(y = f(x) + g(x)\):

| \(x\) | \(-3\) | \(0\) | \(3\) | \(6\) | \(9\) |
| \(f(x) + g(x)\) | 8 | 9 | 7 | 20 | 38 |

How are the slopes of \(f\), \(g\), and \(f + g\) related?

The proof follows:

**Proof:**

\[
\frac{d}{dx}(f(x) + g(x)) = \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x)
\]

\[\Box\]

The same argument applies to \(\frac{d}{dx}(f(x) - g(x))\) so in general we say \(\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)\).

3. You might wonder if the same is true of the product of two functions, \(\frac{d}{dx}(f(x) \cdot g(x))\). As before, it’s worth some time to explore intuitively first so reuse the tables above to see if \(\frac{d}{dx}(f(x) \cdot g(x)) \approx f'(x) \cdot g'(x)\).

| \(x\) | \(-3\) | \(0\) | \(3\) | \(6\) | \(9\) |
| \(f(x)\) | 10 | 8 | 1 | 5 | 11 |
| \(g(x)\) | -2 | 1 | 6 | 15 | 27 |
| \(f(x) \cdot g(x)\) | -20 | 8 | 6 | 75 | 297 |

What does this suggest is (or isn’t) true about \(\frac{d}{dx}(f(x) \cdot g(x))\)?

4. **Power Rule Examples:**

   (a) \(\frac{d}{dx}(5x^2)\)

   (b) \(\frac{d}{dx}(\pi x^5 - \frac{2}{x^3})\)

   (c) \(\frac{d}{dx}\left(\frac{1}{x} - \frac{3\sqrt{x}}{x^3}\right)\)

   (d) \(\frac{d}{dx}\left(\frac{x^4 - 7x}{x^2}\right)\)