Problems

Show all relevant work!

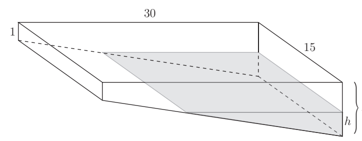
You may use a calculator to verify solutions, but not to provide them.

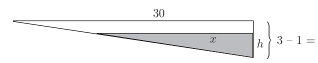
1. 900 cc/sec

Hint: $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 3$ cm/sec., want $\dfrac{\mathrm{d}V}{\mathrm{d}t}$, and we know $V = x^3$ so $\dfrac{\mathrm{d}V}{\mathrm{d}x} = 3x^2$

2. $\frac{1}{30}$ m/min or $\frac{10}{3}$ cm/min.

Hint: Want $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ and we know $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 15$ m^3/min.
The volume of the pool for $h \leq 2$ is $V = \frac{1}{2}15 \cdot h \cdot x$.
From the triangle, we have $\frac{x}{h} = \frac{30}{2} \longrightarrow x = 15h$.
So $V = \dfrac{225}{2}h^2$ and $\dfrac{\mathrm{d}V}{\mathrm{d}h} = 225h \longrightarrow = 450$ at $h = 2$.





3. Draining at $\frac{49\pi}{36}$ ft/min.

Hint: Want $\dfrac{\mathrm{d}V}{\mathrm{d}t}$ and we know $\dfrac{\mathrm{d}h}{\mathrm{d}t} = -0.25$ ft/min.

To relate V and h we know $V = \frac{1}{3}\pi r^2 h$ but we need to eliminate the r so use similar triangles: $\frac{r}{h} = \frac{7}{12} \longrightarrow r = \frac{7}{12}h$.
Then $V = \dfrac{1}{3}\pi\left(\dfrac{7}{12}h\right)^2 h = \dfrac{49\pi}{432}h^3$ and $\dfrac{\mathrm{d}V}{\mathrm{d}h} = \dfrac{49\pi}{144}h^2$.

4. (a) $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{24}{\pi}$ in/sec.

Hint: Want $\dfrac{\mathrm{d}P}{\mathrm{d}t}$ and we know $\dfrac{\mathrm{d}C}{\mathrm{d}t} = 6$ in/sec. To relate P and C note that the side of the square is equal to $2r$ so $P = 8r$. Then since $C = 2\pi r \longrightarrow r = \dfrac{C}{2\pi}$ So $P = 8\left(\dfrac{C}{2\pi}\right) = \dfrac{4C}{\pi}$. It follows $\dfrac{\mathrm{d}P}{\mathrm{d}C} = \dfrac{4}{\pi}$.

(b) $\frac{\mathrm{d}A}{\mathrm{d}t} = 120\left(\frac{1}{\pi} - \frac{1}{4}\right)$ in^2/sec

Hint: Want $\dfrac{\mathrm{d}A}{\mathrm{d}t}$ and we know $\dfrac{\mathrm{d}C}{\mathrm{d}t} = 6$ in/sec. To relate A and C note that the side of the square is equal to $2r$ so $A_\square = (2r)^2$. Then since $C = 2\pi r \longrightarrow r = \dfrac{C}{2\pi}$ So $A_\square = \left(2 \cdot \dfrac{C}{2\pi}\right)^2 = \dfrac{C^2}{\pi^2}$. Then $A = \dfrac{C^2}{\pi^2} - \dfrac{C^2}{4\pi}$.

It follows $\dfrac{\mathrm{d}A}{\mathrm{d}C} = 2C\left(\dfrac{1}{\pi^2} - \dfrac{1}{4\pi}\right)$. When $A = 25\pi \to r = 5$ so $C = 10\pi$.

5. $\frac{10}{3}$ cm^2/sec.

Hint: Want $\dfrac{\mathrm{d}A}{\mathrm{d}t}$ and we know $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 10$ cc/sec. To relate A and V note that both are given in terms of r:
$A = 4\pi r^2$ and $V = \dfrac{4}{3}\pi r^3$ so $V = \dfrac{4}{3}\pi\left(\dfrac{A}{4\pi}\right)^{3/2}$. It follows $\dfrac{\mathrm{d}V}{\mathrm{d}A} = \dfrac{1}{2}\left(\dfrac{A}{4\pi}\right)^{1/2}$.

6. 800m of the \$1/m fence and 200m of the \$2/m fence. The cost will be \$1200.

Hint: If x is the front and y represents the sides, we have (1) $x \cdot y = 60000$ and (2) $C = 2x + 1y + 1x + 1y$.
Simplifying and substituing (1) we have $C = 3x + 2\left(\dfrac{60000}{x}\right)$.

7. $r \approx 3.56$cm and $h \approx 8.91$cm.

Hint: Want to minimize cost through surface area: $C = (0.03)\pi r^2 + (0.02)2\pi rh + (0.02)\pi r^2 = (0.05)\pi r^2 + 0.04\pi rh$.
The additional constraint is $V = \pi r^2 h = 355$. Substituting the constraint for h gives $C = (0.05)\pi r^2 + 0.04\pi r\left(\dfrac{355}{\pi r^2}\right)$.
It follows we need to maximize $C = (0.05)\pi r^2 + \dfrac{14.2}{r}$.

8. (a) When $x = 1$, $A = \frac{1}{2}$.

Hint: Area of rectangle is given by base (x) and height ($y = \frac{1}{x^2+1}$) so $A = \dfrac{x}{x^2+1}$. Find max. of this function.

(b) No. IP at $x = \sqrt{\frac{1}{3}}$

Hint: Want y'' (NOT A''). Note $y'' = \dfrac{2(3x^2-1)}{(x^2+1)^3}$.

9. $r = \frac{20\sqrt{2}}{3}$cm, $h = \frac{40}{3}$cm.

Hint: If r is the radius of the base of the cone, then using the radius of the sphere = 10cm and Pythagoras, we have the height of the cone, $h = 10 + \sqrt{10^2 - r^2}$, assuming the cone extends below the equator.

Note: If $h < 10$, (so the base of the cone is above the equator) then we have $h = 10 - \sqrt{10^2 - r^2}$. Can you see why there is a cone with the same base area but $h > 10$ so we can ignore all cones where $h < 10$?

From this it follows $V = \dfrac{1}{3}\pi r^2(10 + \sqrt{10^2 - r^2})$ BUT this produces a really unpleasant derivative.

Consider instead writing V as a function of h. From above you should see that $r^2 = 20h - h^2$. Use this to help.

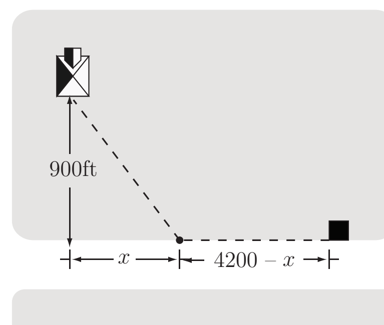
10. We want to maximize $A = xy$ with the constraint that $2x + 2y = k$. $2x + 2y = k \longrightarrow y = \frac{k-2x}{2}$.

Then $A = x\left(\frac{k-2x}{2}\right) = \frac{1}{2}kx - x^2$. So $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{k}{2} - 2x = 0 \longrightarrow x = \frac{k}{4}$. Since $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -2$, this is a maximum.

It follows that since $y = \frac{k-2(\frac{k}{4})}{2} = \frac{k}{4}$, the rectangle is a square.

11. (a) 3573.3′ from the box along the road.

Hint: Want to minimize cost: $C = 35\sqrt{900^2 + x^2} + 20(4200 - x)$. Notice the domain bounds: [0, 4200] and what they mean in this context. $x = 4200$ would be the shortest overall *distance*, while $x = 0$ would be the shortest path through the forest.



(b) \$109,850.53 (Note that $C(0) = \$115,500$)

(c) \$150,337.12 (when $x = 4200$)

12. 36π

13. $\frac{\pi}{4}$

14. (a) 20′ for circle and 0′ for square.

(b) $\frac{20\pi}{\pi+4}'$ for circle and $\frac{80}{\pi+4}'$ for square.

15. \$37