Practice Problems Solutions

Show all relevant work!

1. Suppose $h(x) = (f(x))^3$. If $f(1) = -2$ and $f'(1) = 5$, find $h'(1)$.

Solution: $h'(x) = 3(f(x)^2 \cdot f'(x)$ so $h'(1) = 3(f(1)^2 \cdot f'(1) = 3(-2)^2 \cdot 5 = 60$

□

2. The half-life of of Pu-238 is 88 years. Let Q_0 represent the initial quantity of Pu-238 and assume the decay of the element is continuous.

(a) Using base e, write the particular equation giving the quantity, Q, in grams as a function of time, t, in years.

Solution: We begin with $Q(t) = Q_0 e^{kt}$. Since half of Q_0 remains after 88 years we write $\frac{1}{2}Q_0 = Q_0 e^{88k}$ It follows that $\frac{1}{2} = e^{88k}$ so $k = \frac{1}{88}\ln\left(\frac{1}{2}\right) \approx -0.007877$. Therefore $Q(t) = Q_0 e^{-0.007877t}$

□

(b) Find $\frac{\mathrm{d}Q}{\mathrm{d}t}$ and explain what it represents. Include units in your explanation.

Solution: $\frac{\mathrm{d}Q}{\mathrm{d}t} = -0.007877 Q_0 e^{-0.007877t}$ grams/year. This is the rate at which the quantity (in grams) of Pu-238 is decaying at any time, t.

□

(c) What is the annual rate of decay from your model in (a)?
Compare and contrast this rate with your answer to part (b).

Solution: $e^{-0.007877} \approx 0.9922$ so the annual rate of decay is roughly $1 - 0.9922 = 0.0078$ or about 0.78% per year. The derivative with respect to time takes the percentage of the quantity present at time t and gives the decay rate in terms of actual quantity lost per year. The annual percent gives the relative change in quantity.

3. The daily cost, C, of running an air conditioner in Arizona depends on the temperature, H, as shown in the first table. The temperature in turn increases with the time of day, t, as shown in the second table. Determine the rate at which cost changes with time when $t = 10$ and interpret the result.

H (in F°)	90	95	100	105	110
$C(H)$ (\$)	4	4.75	6	7.50	9.15

t (in hours past 00:00)	6	8	10	12	14
$H(t)$ (in F°)	90	97	100	112	119

Solution: We want to find $\frac{\mathrm{d}}{\mathrm{d}t}C(H(t))$ at $t = 10$. From the Chain Rule $\frac{\mathrm{d}}{\mathrm{d}t}C(H(t)) = C'(H(t)) \cdot H'(t)$ and our solution follows from $C'(H(10)) \cdot H'(10)$.
$C'(H(10)) = C'(100) \approx \frac{7.50-4.75}{105-95} = 0.275$. $H'(10) \approx \frac{112-97}{12-8} = 3.75$
Therefore $C'(H(10)) \cdot H'(10) \approx (0.275)(3.75) \approx 1.03$.

So at 10:00 am, the cost of running an airconditioner in Arizona is increasing at a rate of \$1.03 per hour.

□

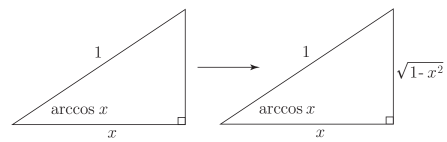
4. Suppose $m'(x) = \frac{x^3}{\sqrt{1-x^4}}$. What is a possible formula for $m(x)$?

Solution: From our experience with the derivative of $y = \sqrt{x}$, we might guess that our solution is something like $y = \sqrt{1-x^4}$. From this we get $y' = \frac{1}{2} \cdot \frac{-4x^3}{\sqrt{1-x^4}} = \frac{-2x^3}{\sqrt{1-x^4}}$. Since we want $m'(x) = \frac{x^3}{\sqrt{1-x^4}}$, we need to eliminate the factor of -2. So if we go back to our original guess of $y = \sqrt{1-x^4}$ and append the multiplicative inverse of -2, we get $m(x) = \frac{-1}{2}\sqrt{1-x^4}$.

□

5. Show that $\frac{\mathrm{d}}{\mathrm{d}x} \arccos x = \frac{-1}{\sqrt{1-x^2}}$

Solution: As with other inverse functions we begin by writing
$\cos(\arccos x) = x$ and differentiating both sides gives
$\frac{\mathrm{d}}{\mathrm{d}x} \cos(\arccos x) = \frac{\mathrm{d}}{\mathrm{d}x} x \longrightarrow$
$-\sin(\arccos x) \cdot \frac{\mathrm{d}}{\mathrm{d}x} \arccos x = 1$
$\frac{\mathrm{d}}{\mathrm{d}x} \arccos x = \frac{1}{-\sin(\arccos x)}$ and from the diagram,
this yields $\frac{\mathrm{d}}{\mathrm{d}x} \arccos x = \frac{-1}{\sqrt{1-x^2}}$.



Note that the diagram implies $0 \le \arccos x \le \frac{\pi}{2}$ whereas the range of $f(x) = \arccos x$ is $[0, \pi]$. Why is $[0, \frac{\pi}{2}]$ sufficient?

□

6. Use the table below to help you find $\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x)$ evaluated at $x = 4$.

x	0	1	2	3	4
$f(x)$	1	2	4	5	5.6
$f'(x)$	1	$\frac{9}{5}$	$\frac{3}{2}$	$\frac{4}{5}$	$\frac{1}{3}$

Solution: In general we find the derivative function for $f^{-1}(x)$ as follows:
$f(f^{-1}(x)) = x \longrightarrow \frac{\mathrm{d}}{\mathrm{d}x} f(f^{-1}(x)) = \frac{\mathrm{d}}{\mathrm{d}x} x$
$f'(f^{-1}(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = 1$
$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$
From this we evaluate $\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x)$ at $x = 4$:
$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x)\Big|_{x=4} = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{3/2} = \frac{2}{3}$

□

7. Show $\frac{\mathrm{d}}{\mathrm{d}x} b^x = \ln(b) b^x$.

Solution: Since we know $\frac{\mathrm{d}}{\mathrm{d}x} e^{kx} = ke^{kx}$ it follows that since $b^x = (e^{\ln b})^x$ we have
$\frac{\mathrm{d}}{\mathrm{d}x} b^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{(\ln b)x} = e^{(\ln b)x} \cdot \ln b = b^x \cdot \ln b$.

□

8. Find $\frac{\mathrm{d}}{\mathrm{d}x} \log_b x$.

Solution: As with most proofs involving inverse functions we begin with $b^{\log_b x} = x$.
Taking derivative of both sides produces $\frac{\mathrm{d}}{\mathrm{d}x} b^{\log_b x} = \frac{\mathrm{d}}{\mathrm{d}x} x$
$\Rightarrow (\ln b) b^{\log_b x} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \log_b x = 1$
$\Rightarrow (\ln b) x \cdot \frac{\mathrm{d}}{\mathrm{d}x} \log_b x = 1$
$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x} \log_b x = \frac{1}{(\ln b)x}$

□

Note the similarity between $\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$ and $\frac{\mathrm{d}}{\mathrm{d}x} \log_b x = \frac{1}{(\ln b)x}$. As with the derivative of exponential functions, they differ only by a factor of $\ln b$ (or $\frac{1}{\ln b}$ in this case).

9. Where is $y = \arctan(1 - x^2)$ increasing and where is it decreasing? Where is it concave up?

Solution: From the Chain Rule we have $f'(x) = \frac{-2x}{1+(1-x^2)^2}$. The only place $f'(x) = 0$ is at $x = 0$.
A quick check of values below and above 0 shows that $f'(-1) > 0$ while $f'(1) < 0$.
Therefore $f(x)$ is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$.

From the quotient Rule and Chain Rule we get
$$f''(x) = \frac{-2(1 + (1-x^2)^2) - (-2x) \cdot 2(1-x^2)(-2x)}{[1 + (1-x^2)^2]^2} = \frac{6x^4 - 4x^2 - 4}{[1 + (1-x^2)^2]^2} = \frac{2(3x^4 - 2x^2 - 2)}{[1 + (1-x^2)^2]^2}$$
Setting the numerator equal to zero gives us $2(3x^4 - 2x^2 - 2) = 0$. Ignoring the 2 and letting $u = x^2$ produces $3u^2 - 2u - 2 = 0$ and the QF yields $u = \frac{1+\sqrt{7}}{3}$; therefore $x = \pm\sqrt{\frac{1+\sqrt{7}}{3}}$. Testing yields:

$x < -\sqrt{\frac{1+\sqrt{7}}{3}}$	$-\sqrt{\frac{1+\sqrt{7}}{3}} < x < \sqrt{\frac{1+\sqrt{7}}{3}}$	$x > \sqrt{\frac{1+\sqrt{7}}{3}}$
$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$

It follows that $y = \arctan(1 - x^2)$ is concave up on (roughly) $(-\infty, -1.102)$ and $(1.102, \infty)$ and concave down on $(-1.102), 1.102)$.

□