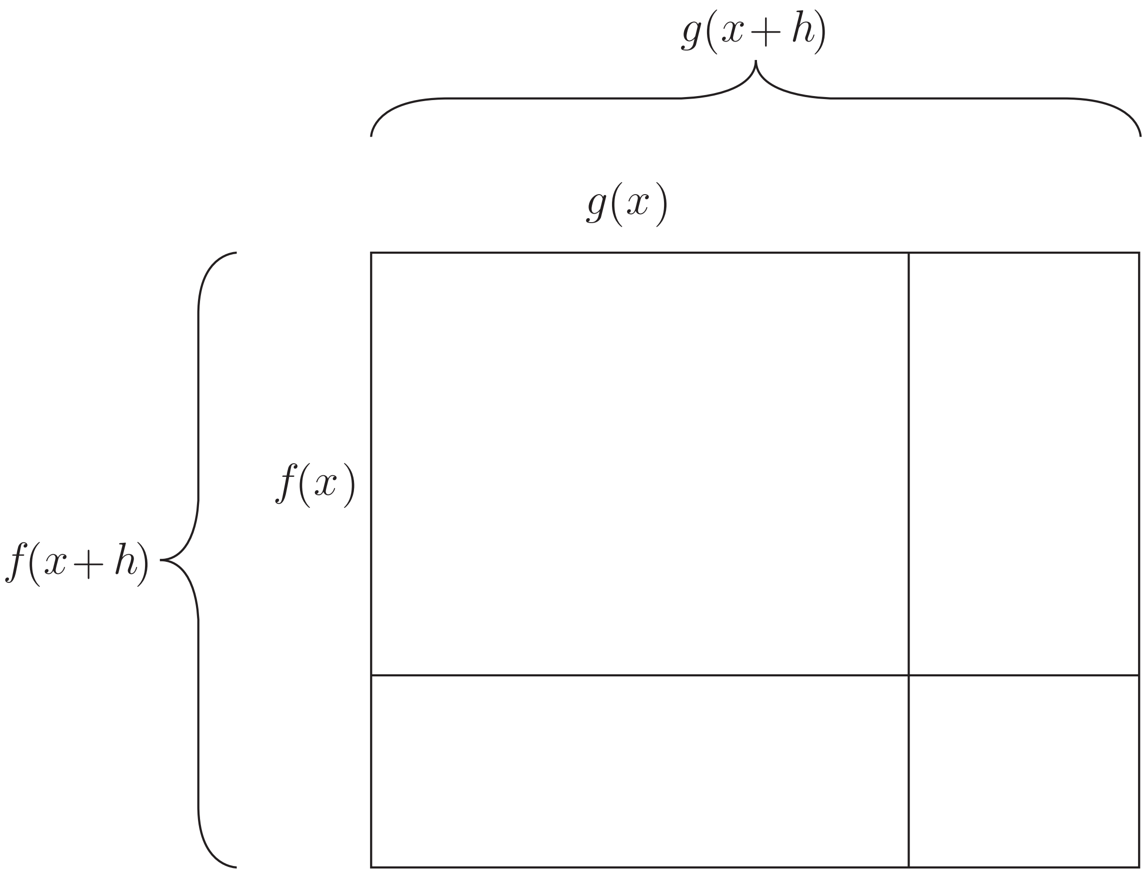
The derivative of $f(x) \cdot g(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \cdot g(x)\right) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \tag{1}$$

Now consider the geometric representation of the expression $f(x+h)g(x+h) - f(x)g(x)$:
If f and g are increasing functions then we have:



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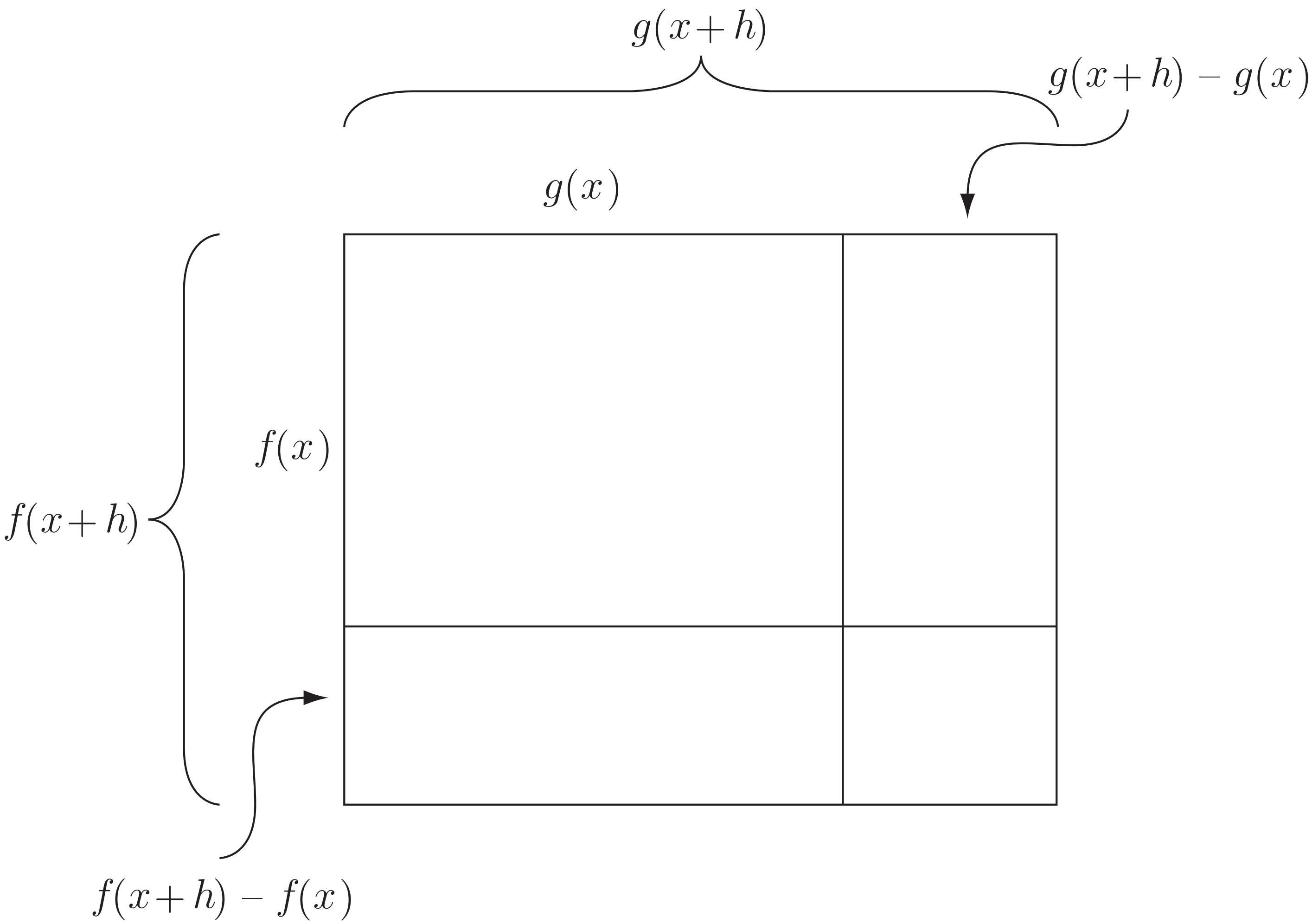

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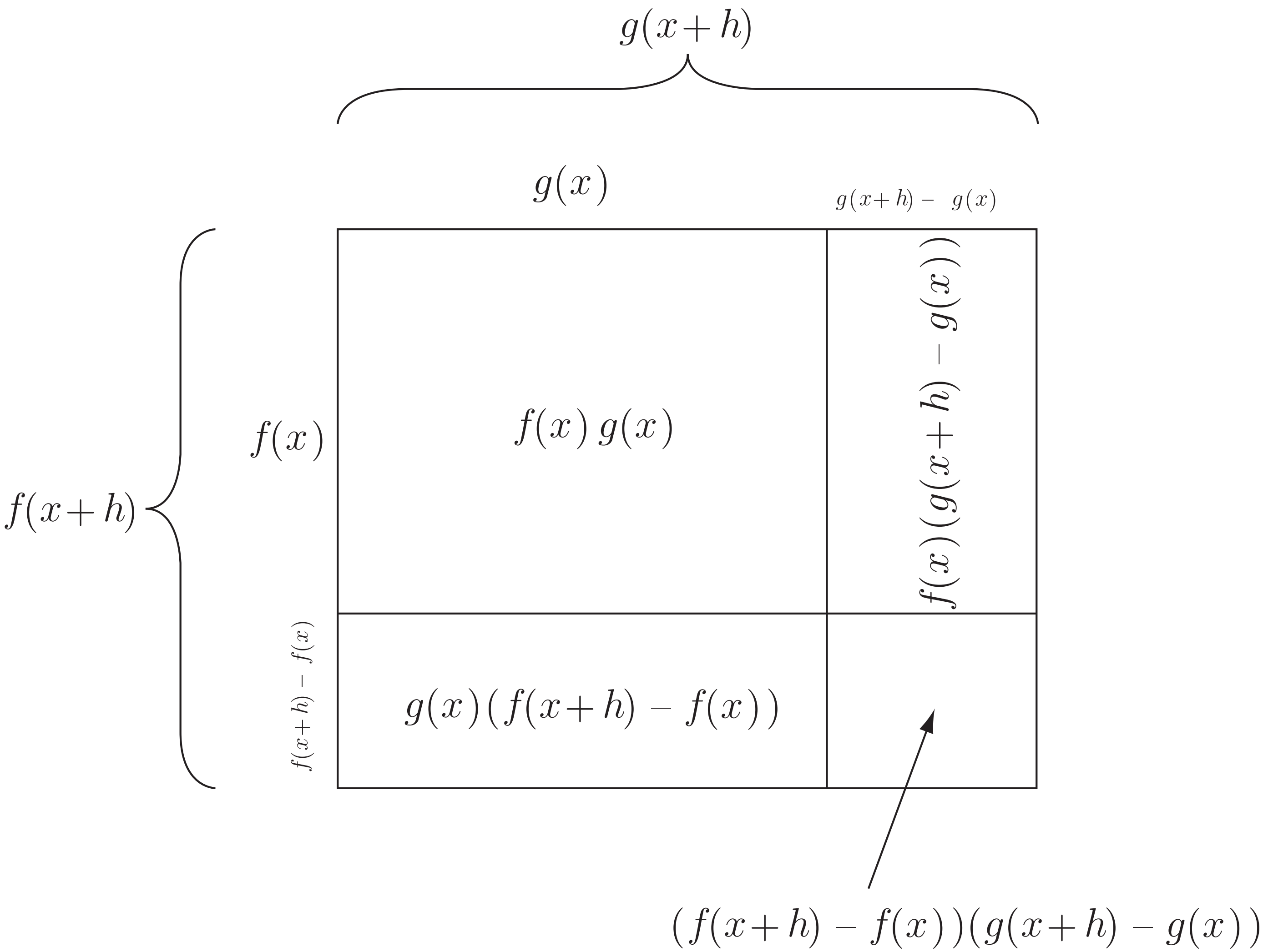

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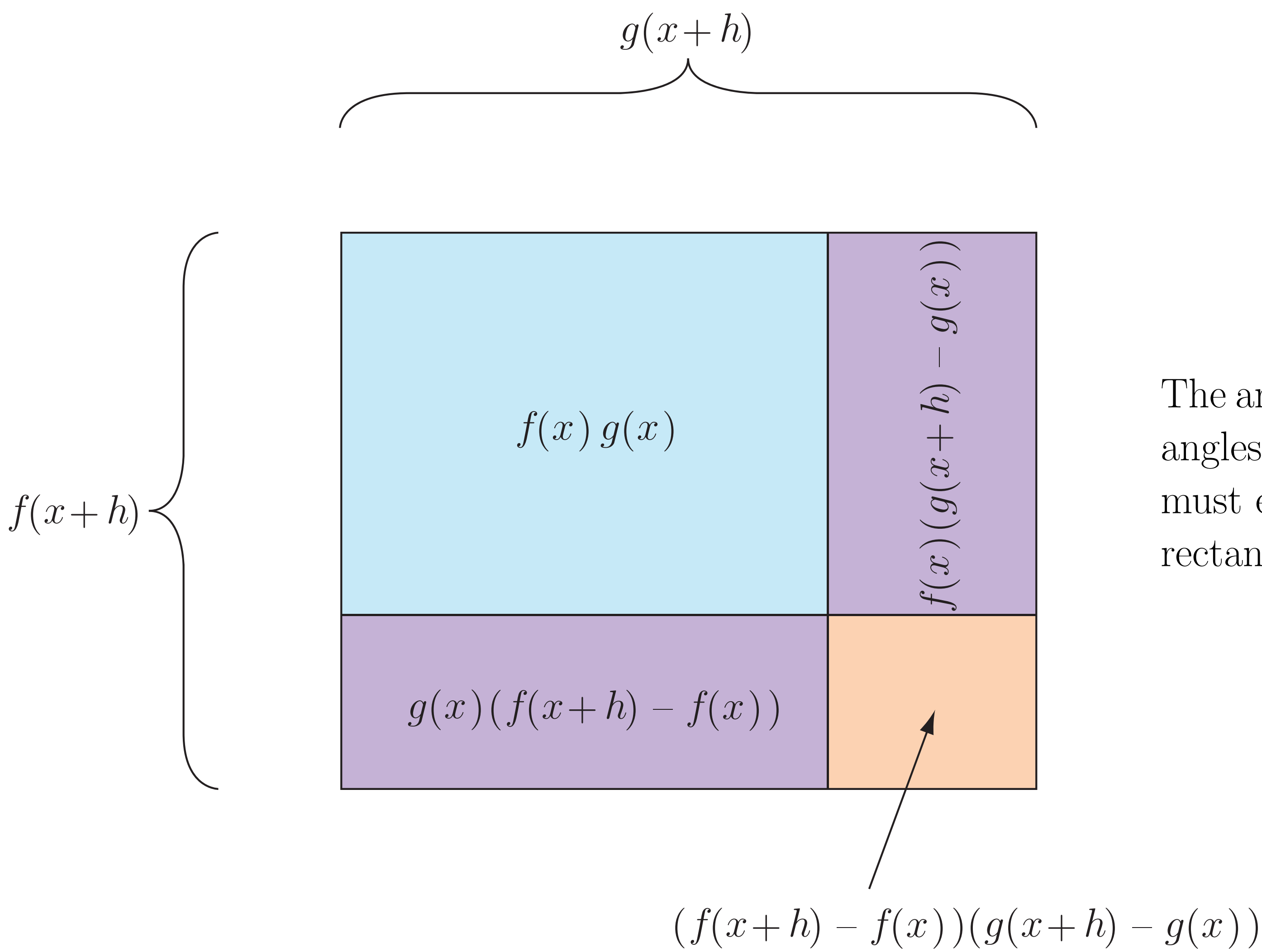
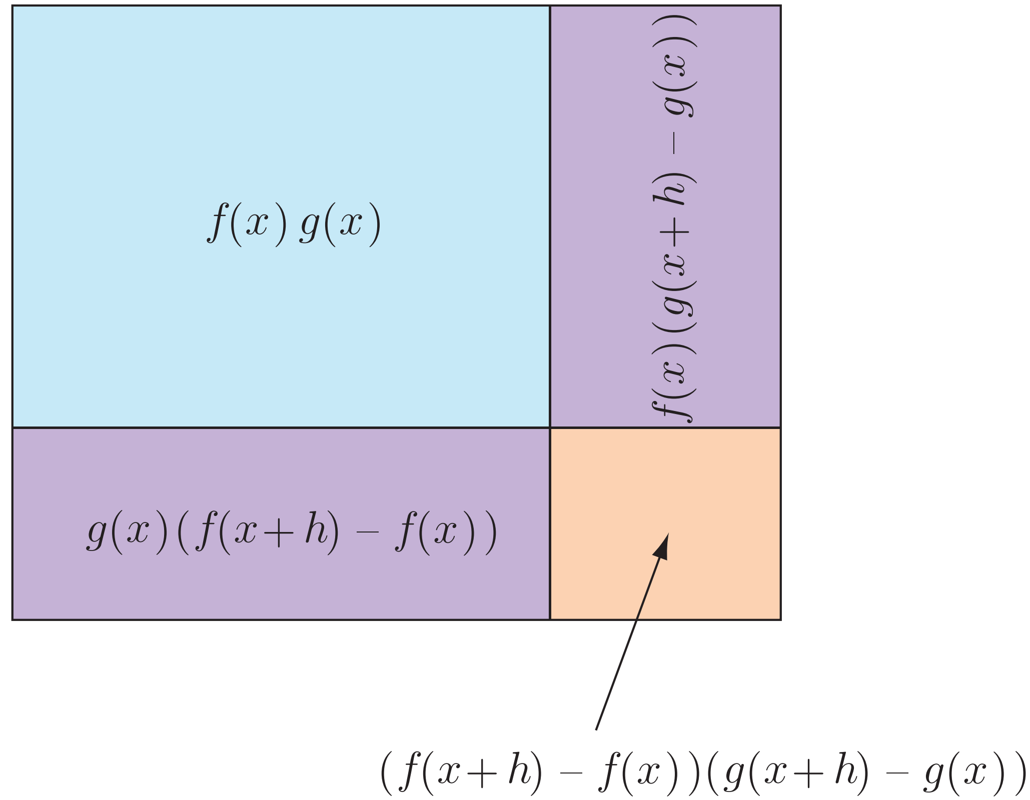


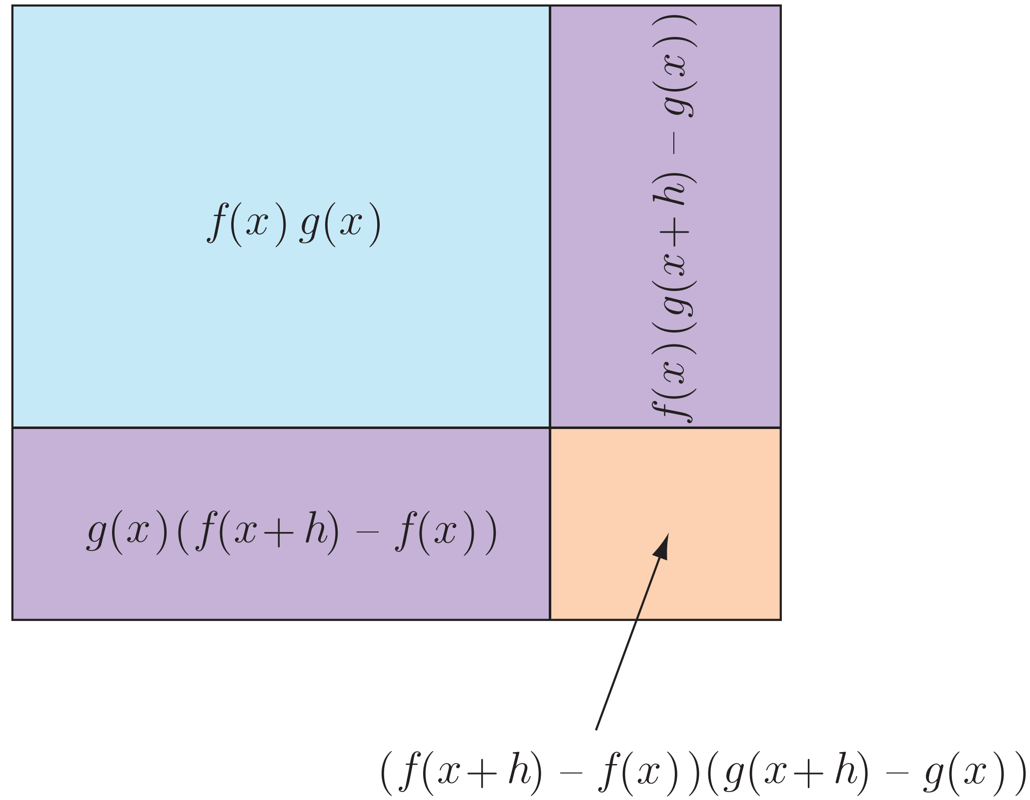
The areas of the four smaller rectangles are shown. Their sum must equal the area of the entire rectangle: $f(x+h)g(x+h)$.

From the diagram we have the following equivalence:



$$f(x+h)g(x+h) = f(x)g(x) + g(x)(f(x+h) - f(x)) + f(x)(g(x+h) - g(x)) + (f(x+h) - f(x))(g(x+h) - g(x)) \tag{2}$$

From the diagram we have the following equivalence:



$$f(x+h)g(x+h) = f(x)g(x) + g(x)(f(x+h) - f(x)) + f(x)(g(x+h) - g(x)) + (f(x+h) - f(x))(g(x+h) - g(x)) \quad (3)$$

$$f(x+h)g(x+h) - f(x)g(x) = g(x)(f(x+h) - f(x)) + f(x)(g(x+h) - g(x)) + (f(x+h) - f(x))(g(x+h) - g(x)) \quad (4)$$

Returning to our original derivative we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right)=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \tag{5}$$

Returning to our original derivative we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \tag{6}$$

$$= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))+f(x)(g(x+h)-g(x))+(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{7}$$

Returning to our original derivative we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \tag{8}$$

$$= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))+f(x)(g(x+h)-g(x))+(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{9}$$

$$= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))}{h}+\lim_{h\to 0}\frac{f(x)(g(x+h)-g(x))}{h}+\lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{10}$$

Returning to our original derivative we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \tag{11}$$

$$= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))+f(x)(g(x+h)-g(x))+(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{12}$$

$$= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))}{h}+\lim_{h\to 0}\frac{f(x)(g(x+h)-g(x))}{h}+\lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{13}$$

$$= \qquad f'(x)g(x) \qquad + \qquad f(x)g'(x) \qquad + \lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{14}$$

Returning to our original derivative we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \tag{15}$$

$$= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))+f(x)(g(x+h)-g(x))+(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{16}$$

$$= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))}{h}+\lim_{h\to 0}\frac{f(x)(g(x+h)-g(x))}{h}+\lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{17}$$

$$= f'(x)g(x) + f(x)g'(x) + \lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{18}$$

The last term may be written:

$$\lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))}{h} = \lim_{h\to 0}\frac{(f(x+h)-f(x))}{h}\lim_{h\to 0}(g(x+h)-g(x)) \tag{19}$$

Returning to our original derivative we have

$$
\begin{align}
\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right) &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \tag{20}\\
&= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))+f(x)(g(x+h)-g(x))+(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{21}\\
&= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))}{h}+\lim_{h\to 0}\frac{f(x)(g(x+h)-g(x))}{h}+\lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{22}\\
&= \qquad f'(x)g(x) \qquad + \qquad f(x)g'(x) \qquad + \lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{23}
\end{align}
$$

The last term may be written:

$$
\begin{align}
\lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))}{h} &= \lim_{h\to 0}\frac{(f(x+h)-f(x))}{h}\lim_{h\to 0}(g(x+h)-g(x)) \tag{24}\\
&= \qquad f'(x)\cdot 0 \tag{25}
\end{align}
$$

Returning to our original derivative we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \tag{26}$$

$$= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))+f(x)(g(x+h)-g(x))+(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{27}$$

$$= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))}{h}+\lim_{h\to 0}\frac{f(x)(g(x+h)-g(x))}{h}+\lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{28}$$

$$= f'(x)g(x) + f(x)g'(x) + \lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))}{h} \tag{29}$$

The last term may be written:

$$\lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))}{h} = \lim_{h\to 0}\frac{(f(x+h)-f(x))}{h}\lim_{h\to 0}(g(x+h)-g(x)) \tag{30}$$

$$= f'(x)\cdot 0 \tag{31}$$

So

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right) = f'(x)g(x)+f(x)g'(x)$$