Theorems, Derivatives   Name: ____________________

Mean Value Theorem
If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b),
then there is at least one value, $x = c$, where $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(1) Sketch a graph representing the hypothesis and conclusion of this theorem.

(2) In essence the Mean Value Theorem (MVT) asserts that a vehicle that averages v mph on a trip will at some time on the trip have been traveling at v mph. Give another real-world example of the MVT.

(3) Sketch a graph or give an example of a function that is continuous on $[a, b]$ but does not satisfy the conclusion of the MVT (Consider the hypotheses of the theorem). Why does it fail?

(4) State the *converse* of the Mean Value Theorem and draw or give a counterexample showing it is not true.

Consequences of the Mean Value Theorem:

Increasing Function Theorem
Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then

- If $f'(x) > 0$ on $a < x < b$, then $f(x)$ is increasing on $a \leq x \leq b$.
- If $f'(x) \geq 0$ on $a < x < b$, then $f(x)$ is nondecreasing on $a \leq x \leq b$.

(5) An example of the increasing function theorem would be to say that a car experiencing positive acceleration will be speeding up. Give another real-world example of the Increasing Function Theorem.

(6) State the converse of the IFT (assume $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b)) and provide a counterexample or drawing showing it is not true.

Constant Function Theorem
Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then If $f'(x) = 0$ on $a < x < b$, then $f(x)$ is constant on $a \leq x \leq b$.

(7) State the converse of the CFT (assume $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b)). Prove that the converse is true.

(8) Give an example of a discontinuous function that is not constant (show why we need the requirement that $f(x)$ is continuous).

The Racetrack Principle
Suppose $f(x)$ and $g(x)$ are continuous on $[a, b]$ and differentiable on (a, b).
Also suppose that $f'(x) \leq g'(x)$ on (a, b). Then

- If $f(a) = g(a)$, then $f(x) \leq g(x)$ for $a \leq x \leq b$.
- If $f(b) = g(b)$, then $f(x) \geq g(x)$ for $a \leq x \leq b$.

(9) Sketch graphs illustrating the hypothesis and conclusion(s) of the Racetrack Principle.

(10) Use an example to illustrate the racetrack principle (both parts).

(11) Is it true that for continuous (on $[a, b]$), differentiable (on (a, b)) functions f and g, if $f(x) \leq g(x)$ and $f(a) = g(a)$, then $f'(x) \leq g'(x)$ on (a, b)? Please support your answer.

(12) Show the tangent line approximation for $y = \ln x$ at $x = 1$ is $y = x - 1$ and use the Racetrack Principle to show that $x - 1 \geq \ln x$ for all x in the domain of $y = \ln x$.

Equal Derivatives
Suppose $f(x)$ is differentiable for all x in the domain of f.
If $f(x) = g(x) + C$, $C \in \mathbb{R}$,
Then $f'(x) = g'(x)$

You should see that this is a consequence of the addition property of derivatives.

(13) State the *converse* of the theorem above. Do you think it is true? Please provide some reflections.