1. If a fireman leans a 24 foot ladder against a building at a 70° angle,

(a) How far from the building is the base of the ladder?

Solution: We want the distance from the building (labeled \(x\)). We know the length of the ladder is 24' so relative to the 70° angle we know the hypotenuse and we want the side adjacent so we use cosine:

\[
\cos 70° = \frac{x}{24} \implies 24 \cos 70° \approx 8.2'.
\]

(b) How high (above the ground) does the ladder reach on the building?

Solution: We want the distance from the building (labeled \(y\)). We know the length of the ladder is 24' so relative to the 70° angle we know the hypotenuse and we want the side opposite so we use sine:

\[
\sin 70° = \frac{y}{24} \implies 24 \sin 70° \approx 22.6'.
\]

2. A merry-go-round with a 10 ft. diameter is spinning at 12rpm.

(a) What is the angular velocity of the merry-go-round in radians per second?

Solution: Since there are 2\(\pi\) radians in one full revolution, the merry-go-round is rotating at 12 \cdot 2\(\pi\) = 24\(\pi\) radians per minute. Then this is the same as 24\(\pi\) rad minute \cdot \frac{1}{60} seconds = \(\frac{2\pi}{5}\) rad per second.

(b) How fast (in feet per second) is Raul traveling if he sits on the outer edge of the merry-go-round?

Solution: Since speed is measured in distance traveled over time, we need to know how far Raul travels in a second or minute. Since he makes 12 full revolutions in a minute and the radius of the merry-go-round is 5ft, he's covering 12 \cdot 2\(\pi\)(5) = 120\(\pi\) feet per minute or equivalently \(\frac{120\pi}{60}\) = 2\(\pi\) ft/sec.

(c) How fast is Klaus travelling (in feet per second) if he sits at the center of the merry-go-round?

Solution: Since Klaus is at the center his distance from the center is 0 and therefore isn’t going anywhere. While his angular velocity is \(\frac{2\pi}{5}\) rad per second, his speed is 0.

3. The functions \(\sin x\) and \(\cos x\) are almost identical except for horizontal position (see below). That means you should be able to express \(\sin x\) as a shift of \(\cos x\) and similarly, \(\cos x\) as a shift of \(\sin x\). Specifically, find \(c\) so that \(\sin x = \cos(x + c)\) and find \(c\) so that \(\cos x = \sin(x + c)\).

Solution: In both cases we note that cosine reaches its maximum at \(x = 0\), that is \(\cos(0) = 1\). Meanwhile sine doesn’t reach its maximum until \(\frac{\pi}{2}\) radians later. That means in order to get cosine shifted ahead to sine we have to shift it forward by \(\frac{\pi}{2}\) radians. Therefore, \(\sin x = \cos(x - \frac{\pi}{2})\).

Similarly, for sine to catch up with cosine, it has to be shifted ahead by \(\frac{\pi}{2}\) radians so \(\cos x = \sin(x + \frac{\pi}{2})\).

4. Solve the following for \(x \in \mathbb{R}\).

(a) \(\sin x = 1\)

Solution: On the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\) (this is the interval we restrict sine to in order to make arcsin a function) we know \(\sin x = 1\) when \(x = \arcsin(1) = \frac{\pi}{2}\). Since this is the only occurrence on that interval and sine has a period of \(2\pi\), the general solution is \(x = \frac{\pi}{2} + 2\pi n\), where \(n \in \mathbb{Z}\).

(b) \(\cos 3x = 1\)

Solution: Similar logic here, cosine is usually restricted to the interval \([0, \pi]\) in order to invert it. Then \(3x = \arccos(1) \implies x = 0\). Since \(\cos(3x)\) has a period of \(\frac{2\pi}{3}\), the general solution is \(x = 0 + \frac{2\pi n}{3}\) \(n \in \mathbb{Z}\).
(c) \sin(x^2 - 1) = 1

Solution: From part (a) we know the solution to \sin(x) = 1 is \(x = \frac{\pi}{2} + 2\pi n, n \in \mathbb{Z}\). In this case \(x^2 - 1 = \frac{\pi}{2} + 2\pi n \in \mathbb{Z}\). Solving gives us \(x = \pm \sqrt{1 + \frac{\pi}{2} + 2\pi n}, n \in \mathbb{Z}_{\geq 0}\). Notice the change from \(n \in \mathbb{Z}\) to \(n\) in the non-negative integers since the square root has a restricted domain.

5. Find two different equations for the periodic function shown on the right. (One in terms of \sin x\) and the other in terms of \cos x\).

\[
\begin{align*}
\text{Solution:} & \\
& \text{We need to find the constants } A, B, C \text{ and } D \\
& \text{in the form } y = A \sin(Bx + D) + C \\
& C = \frac{-3 + 5}{2} = 3 \\
& A = 7 - 3 = 4 \\
& \text{The span from peak to peak is } 4\pi \text{ (or the span from top to bottom is } 2\pi) \text{ so the period is } 4\pi \text{ and therefore } B = \frac{1}{2}. \\
& \text{Finally, since the maximum occurs at } \pi \text{ when for sine it normally occurs at } \frac{\pi}{2}, \text{ we need to find } C \text{ so that} \\
& \frac{1}{2}(\pi) + D = \pi \rightarrow D = 0. \\
& \text{This gives us } y = 4 \sin \left(\frac{1}{2}x\right) + 3 \quad \square
\end{align*}
\]

\[
\begin{align*}
\text{Similarly, for cosine we have the same } A, B \text{ and } C \text{ so we only need to find } D. \text{ The maximum or cosine occurs at } 0 \text{ so we need } D \text{ so that} \\
& \frac{1}{2}(0) + D = 0 \rightarrow D = -\frac{\pi}{2}. \\
& \text{This gives us } y = 4 \cos \left(\frac{1}{2}x - \frac{\pi}{2}\right) + 3 \quad \square
\end{align*}
\]

6. A mass is suspended at the end of a spring where it hangs 20cm from the ceiling. It is displaced 8cm below its rest position and released. It reaches the point closest to the ceiling (12cm) after 1 second.

Write a periodic model for this situation giving the distance of the mass from the ceiling, \(y\), as a function of time, \(t\).

\[
\begin{align*}
\text{Solution:} & \\
& \text{Again, we need to find the constants } A, B, C \text{ and } D \text{ in the form } y = A \sin(Bx + D) + C. \text{ The amplitude, } A \text{ is the displacement from center so } A = 8 \text{ if we make the center } C = 20. \\
& \text{The period is } 2 \text{ seconds (to return to start) and therefore } B = \pi. \text{ Finally, since the pendulum begins at its maximum displacement we can use cosine (or } -\cos \text{) and therefore } D = 0. \text{ This gives us } y = 8 \cos(\pi t) + 20. \quad \square
\end{align*}
\]

7. The table below shows the US average unemployment rate at the beginning of each year from 1993 – 2003. (Where 1993 is \(t = 0\)).

<table>
<thead>
<tr>
<th>(t) (years)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U(t)) (% unemployment)</td>
<td>7.3</td>
<td>6.6</td>
<td>5.6</td>
<td>5.6</td>
<td>5.3</td>
<td>4.6</td>
<td>4.3</td>
<td>4.0</td>
<td>4.1</td>
<td>5.6</td>
<td>5.7</td>
</tr>
</tbody>
</table>

a) Write a periodic function that models these data using methods discussed in class.

\[
\begin{align*}
\text{Solution:} & \\
& \text{The maximum unemployment is } 7.3\% \text{ while the minimum is } 4.0\% \text{ so } C = \frac{7.3 + 4}{2} = 5.65. \\
& \text{Then } A = 7.3 - 5.65 = 1.65. \\
& \text{It takes seven years to go from the highest to the lowest value so } 7 \text{ years is half a period and therefore the period is } 14 \text{ years. It follows that } B = \frac{7\pi}{14} = \frac{\pi}{2}. \\
& \text{Finally, if we use cosine which begins at its maximum like the date set, we can ignore } D. \\
& \text{This gives us } U(t) = 1.65 \cos \left(\frac{\pi}{2} t\right) + 5.65 \quad \square
\end{align*}
\]

b) Use your model to predict the unemployment rate for January of 2016.

\[
\begin{align*}
\text{Solution:} & \\
& \text{From the equation in (a) we have } U(13) = 1.65 \cos \left(\frac{\pi}{2} (23)\right) + 5.65 \approx 4.6\% \text{ unemployment.} \quad \square
\end{align*}
\]

(8) Answers: (a) 0.5mi (b) 0.8mi (c) 20.2 mph (d) 2min, 19sec.