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REVIEW EXERCISES AND PROBLEMS FOR CHAPTER 10

EXERCISES

For Exercises 1-4, find the second-degree Taylor polynomial about the given point.

1.
e^x, $x = 1$

ANSWER

WORKED SOLUTION

2.
$\ln x$, $x = 2$

3.
$\sin x$, $x = -\pi/4$

ANSWER

4.
$\tan \theta$, $\theta = \pi/4$

5.
Find the third-degree Taylor polynomial for $f(x) = x^3 + 7x^2 - 5x + 1$ at $x = 1$.

ANSWER

WORKED SOLUTION

For Exercises 6-8, find the Taylor polynomial of degree n for x near the given point a.

6.

$$\frac{1}{1-x}, \quad a = 2, \quad n = 4$$

7.

$$\sqrt{1+x}, \quad a = 1, \quad n = 3$$

ANSWER

8.
$\ln x$, $a = 2$, $n = 4$

9.
Write out P_7, the Taylor polynomial of degree $n = 7$ approximating g near $x = 0$, given that

$$g(x) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1} 3^i}{(i-1)!} x^{2i-1}.$$

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ANSWER

WORKED SOLUTION

10.
Find the first four nonzero terms of the Taylor series around $x = 0$ for $f(x) = \cos^2 x$.
[Hint: $\cos^2 x = 0.5(1 + \cos 2x)$.]

In Exercises 11-18, find the first four nonzero terms of the Taylor series about the origin of the given functions.

11.
$t^2 e^t$

ANSWER

12.
$\cos(3y)$

13.
$\theta^2 \cos \theta^2$

ANSWER

WORKED SOLUTION

14.
$\sin t^2$

15.

$$\frac{t}{1+t}$$

ANSWER

16.

$$\frac{1}{1-4z^2}$$

17.

$$\frac{1}{\sqrt{4-x}}$$

ANSWER
WORKED SOLUTION

18.

$$\frac{z^2}{\sqrt{1-z^2}}$$

For Exercises 19-22, expand the quantity in a Taylor series around 0 in terms of the variable given. Give four nonzero terms.

19.

$$\frac{a}{a+b} \text{ in terms of } \frac{b}{a}$$

ANSWER

20.

$$\frac{1}{(a+r)^{3/2}} \text{ in terms of } \frac{r}{a}$$

21.

$$(B^2+y^2)^{3/2} \text{ in terms of } \frac{y}{B}, \text{where } B>0$$

ANSWER
WORKED SOLUTION

22.

$$\sqrt{R-r} \text{ in terms of } \frac{r}{R}$$

PROBLEMS

23.

A function f has $f(3)=1, f'(3)=5$ and $f''(3)=-10$. Find the best estimate you can for $f(3.1)$.

ANSWER

Find the exact value of the sums in Problems 24-28.

24.

$$3+3+\frac{3}{2!}+\frac{3}{3!}+\frac{3}{4!}+\frac{3}{5!}+\cdots$$

25.

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \cdots$$

ANSWER

WORKED SOLUTION

26.

$$1 - 2 + \frac{4}{2!} - \frac{8}{3!} + \frac{16}{4!} - \cdots$$

27.

$$2 - \frac{8}{3!} + \frac{32}{5!} - \frac{128}{7!} + \cdots$$

ANSWER

28.

$$(0.1)^2 - \frac{(0.1)^4}{3!} + \frac{(0.1)^6}{5!} - \frac{(0.1)^8}{7!} + \cdots$$

29.
Find an exact value for each of the following sums.
(a)
$7(1.02)^3 + 7(1.02)^2 + 7(1.02) + 7 + \frac{7}{(1.02)} + \frac{7}{(1.02)^2} + \cdots + \frac{7}{(1.02)^{100}}$.

ANSWER

WORKED SOLUTION

(b)

$$7 + 7(0.1)^2 + \frac{7(0.1)^4}{2!} + \frac{7(0.1)^6}{3!} + \cdots$$

ANSWER

WORKED SOLUTION

30.
Suppose all the derivatives of some function f exist at 0, and the Taylor series for f about $x = 0$ is

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + \frac{x^n}{n} + \cdots.$$

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Find $f'(0), f''(0), f'''(0)$, and $f^{(10)}(0)$.

31.
Suppose x is positive but very small. Arrange the following expressions in increasing order:

$$x, \quad \sin x, \quad \ln(1+x), \quad 1-\cos x,$$
$$e^x - 1, \quad \arctan x, \quad x\sqrt{1-x}.$$

()
ANSWER

32.
By plotting several of its Taylor polynomials and the function $f(x) = 1/(1+x)$, estimate graphically the interval of convergence of the series expansion for this function about $x = 0$. Compute the radius of convergence analytically.

33.
Find the radius of convergence of the Taylor series around $x = 0$ for $\frac{1}{1-2x}$.
ANSWER
WORKED SOLUTION

34.
Use Taylor series to evaluate $\lim\limits_{x\to 0} \frac{\ln(1+x+x^2)-x}{\sin^2 x}$.

35.
Referring to the table, use a fourth-degree Taylor polynomial to estimate the integral $\int_0^{0.6} f(x)\ dx$.

$f(0)$	$f'(0)$	$f''(0)$	$f'''(0)$	$f^{(4)}(0)$
0	1	−3	7	−15

ANSWER

36.
Let $f(x) = e^{-x^3}$.
(a)
Write the first five nonzero terms of the Taylor series for $f(x)$ centered at $x = 0$.

(b)
Write the first four nonzero terms of the Taylor series for $f'(x)$ centered at $x = 0$.

37.
Use a Taylor polynomial of degree $n = 8$ to estimate $\int_0^1 \cos\left(x^2\right)\ dx$.

ANSWER

WORKED SOLUTION

38.

(a)

Find $\lim_{\theta \to 0} \frac{\sin(2\theta)}{\theta}$. Explain your reasoning.

(b)

Use series to explain why $f(\theta) = \frac{\sin(2\theta)}{\theta}$ looks like a parabola near $\theta = 0$. What is the equation of the parabola?

39.

(a)

Find the Taylor series expansion of arcsin x.

ANSWER

(b)

Use Taylor series to find the limit as $x \to 0$ of $\frac{\arctan x}{\arcsin x}$.

ANSWER

40.

Let $f(0) = 1$ and $f^{(n)}(0) = \frac{(n+1)!}{2^n}$ for $n > 0$.

(a)

Write the Taylor series for f at $x = 0$ using sigma sum notation. Simplify the general term.
[Hint: Write out the first few terms of the Taylor series.]

(b)

Does the series you found in part a converge for $x = 3$? Briefly explain your reasoning.

(c)

Use the series you found in part a to evaluate $\int_0^1 f(x)\ dx$. You may assume that

$$\int_0^1 \left(\sum_{n=1}^{\infty} a_n x^n \right) dx = \sum_{n=1}^{\infty} \left(\int_0^1 a_n x^n \ dx \right).$$

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41.

In this problem, you will investigate the error in the n^{th} degree Taylor approximation to e^x about 0 for various values of n.

(a)

Let $E_1 = e^x - P_1(x) = e^x - (1 + x)$. Using a calculator or computer, graph E_1 for $-0.1 \le x \le 0.1$. What shape is the graph of E_1? Use the graph to confirm that

$$|E_1| \le x^2 \quad \text{for} \quad -0.1 \le x \le 0.1.$$

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ANSWER

WORKED SOLUTION

(b)

Let $E_2 = e^x - P_2(x) = e^x - (1 + x + x^2/2)$. Choose a suitable range and graph E_2 for $-0.1 \le x \le 0.1$. What shape is the graph of E_2? Use the graph to confirm that

$$|E_2| \le x^3 \quad \text{for} \quad -0.1 \le x \le 0.1.$$

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ANSWER

WORKED SOLUTION

(c)

Explain why the graphs of E_1 and E_2 have the shapes they do.

WORKED SOLUTION

42.

The table gives values of $f^{(n)}(0)$ where f is the *inverse hyperbolic tangent* function. Note that $f(0) = 0$.

n	1	2	3	4	5	6	7
$f^{(n)}(0)$	1	0	2!	0	4!	0	6!

(a)

Find the Taylor polynomial of degree 7 for f about $x = 0$.

(b)

Assuming the pattern in the table continues, write the Taylor series for this function.

43.

A particle moving along the x-axis has potential energy at the point x given by $V(x)$. The potential energy has a minimum at $x = 0$.

(a)

Write the Taylor polynomial of degree 2 for V about $x = 0$. What can you say about the signs of the coefficients of each of the terms of the Taylor polynomial?

ANSWER

(b)

The force on the particle at the point x is given by $-V'(x)$. For small x, show that the force on the particle is approximately proportional to its distance from the origin. What is the sign of the proportionality constant? Describe the direction in which the force points.

ANSWER

44.

Consider the functions $y = e^{-x^2}$ and $y = 1/(1 + x^2)$.

(a)

Write the Taylor expansions for the two functions about $x = 0$. What is similar about the two series? What is different?

(b)

Looking at the series, which function do you predict will be greater over the interval $(-1, 1)$? Graph both and see.

(c)

Are these functions even or odd? How might you see this by looking at the series expansions?

(d)

By looking at the coefficients, explain why it is reasonable that the series for $y = e^{-x^2}$ converges for all values of x, but the series for $y = 1/(1 + x^2)$ converges only on $(-1, 1)$.

45.

The Lambert W function has the following Taylor series about $x = 0$:

$$W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n.$$

()

Find P_4, the fourth-degree Taylor polynomial for $W(x)$ about $x = 0$.

ANSWER

WORKED SOLUTION

46.

Using the table, estimate the value of $\int_0^2 f(x)\ dx$.

$f(0)$	$f'(0)$	$f''(0)$	$f'''(0)$	$f^{(4)}(0)$	$f^{(5)}(0)$
2	0	−1	0	−3	6

47.

Let $f(t)$ be the so called *exponential integral*, a special function with applications to heat transfer and water flow,[13] http://en.wikipedia.org/wiki/Exponential_integral. which has the property that

$$f'(t) = t^{-1} e^{t}.$$

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Use the series for e^t about $t = 0$ to show that

$$f(t) \approx \ln t + P_3(t) + C,$$

()

where P_3 is a third-degree polynomial. Find P_3. You need not find the constant C.

ANSWER

48.

The electric potential, V, at a distance R along the axis perpendicular to the center of a charged disc with radius a and constant charge density σ, is given by

$$V = 2\pi\sigma(\sqrt{R^2 + a^2} - R).$$

()

Show that, for large R,

$$V \approx \frac{\pi a^2 \sigma}{R}.$$

()

49.

The *gravitational field* at a point in space is the gravitational force that would be exerted on a unit mass placed there. We will assume that the gravitational field strength at a distance d away from a mass M is

$$\frac{GM}{d^2}$$

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where G is constant. In this problem you will investigate the gravitational field strength, F, exerted by a system consisting of a large mass M and a small mass m, with a distance r between them. (See Figure 10.39.)

(a)

Write an expression for the gravitational field strength, F, at the point P.

ANSWER

WORKED SOLUTION

(b)

Assuming r is small in comparison to R, expand F in a series in r/R.

ANSWER

WORKED SOLUTION

(c)

By discarding terms in $(r/R)^2$ and higher powers, explain why you can view the field as resulting from a single particle of mass $M + m$, plus a correction term. What is the position of the particle of mass $M + m$? Explain the sign of the correction term.

Figure 10.39

WORKED SOLUTION

50.

A thin disk of radius a and mass M lies horizontally; a particle of mass m is at a height h directly above the center of the disk. The gravitational force, F, exerted by the disk on the mass m is given by

$$F = \frac{2GMmh}{a^2}\left(\frac{1}{h} - \frac{1}{(a^2 + h^2)^{1/2}}\right),$$

()

where G is a constant. Assume $a < h$ and think of F as a function of a, with the other quantities constant.

(a)

Expand F as a series in a/h. Give the first two nonzero terms.

(b)

Show that the approximation for F obtained by using only the first nonzero term in the series is independent of the radius, a.

(c)

If $a = 0.02h$, by what percentage does the approximation in part a differ from the approximation in part b?

51.

When a body is near the surface of the earth, we usually assume that the force due to gravity on it is a constant mg, where m is the mass of the body and g is the acceleration due to gravity at sea level. For a body at a distance h above the surface of the earth, a more accurate expression for the force F is

$$F = \frac{mgR^2}{(R + h)^2}$$

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where R is the radius of the earth. We will consider the situation in which the body is close to the surface of the earth so that h is much smaller than R.

(a)

Show that $F \approx mg$.

(b)

Express F as mg multiplied by a series in h/R.

ANSWER

(c)
The first-order correction to the approximation $F \approx mg$ is obtained by taking the linear term in the series but no higher terms. How far above the surface of the earth can you go before the first-order correction changes the estimate $F \approx mg$ by more than 10%? (Assume $R = 6400$ km.)

ANSWER

52.
Expand $f(x + h)$ and $g(x + h)$ in Taylor series and take a limit to confirm the product rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

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53.
Use Taylor expansions for $f(y + k)$ and $g(x + h)$ to confirm the chain rule:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

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WORKED SOLUTION

54.
All the derivatives of g exist at $x = 0$ and g has a critical point at $x = 0$.
(a)
Write the n^{th} Taylor polynomial for g at $x = 0$.

(b)
What does the Second Derivative test for local maxima and minima say?

(c)
Use the Taylor polynomial to explain why the Second Derivative test works.

55.
(Continuation of Problem 54) You may remember that the Second Derivative test tells us nothing when the second derivative is zero at the critical point. In this problem you will investigate that special case. Assume g has the same properties as in Problem 54, and that, in addition, $g''(0) = 0$. What does the Taylor polynomial tell you about whether g has a local maximum or minimum at $x = 0$?

56.
Use the Fourier series for the square wave

$$f(x) = \begin{cases} -1 & -\pi < x \leq 0 \\ 1 & 0 < x \leq \pi \end{cases}$$

()

to explain why the following sum must approach $\pi/4$ as $n \to \infty$:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^n \frac{1}{2n+1}.$$

()

You may assume that the Fourier series converges to $f(x)$ at $x = \pi/2$.

57.

Suppose that $f(x)$ is a differentiable periodic function of period 2π. Assume the Fourier series of f is differentiable term by term.

(a)

If the Fourier coefficients of f are a_k and b_k, show that the Fourier coefficients of its derivative f' are kb_k and $-ka_k$.

WORKED SOLUTION ⊕

(b)

How are the amplitudes of the harmonics of f and f' related?

ANSWER ⊕

WORKED SOLUTION ⊕

(c)

How are the energy spectra of f and f' related?

ANSWER ⊕

WORKED SOLUTION ⊕

58.

If the Fourier coefficients of f are a_k and b_k, and the Fourier coefficients of g are c_k and d_k, and if A and B are real, show that the Fourier coefficients of $Af + Bg$ are $Aa_k + Bc_k$ and $Ab_k + Bd_k$.

59.

Suppose that f is a periodic function of period 2π and that g is a horizontal shift of f, say $g(x) = f(x + c)$. Show that f and g have the same energy.

CAS Challenge Problems

60.

(a)

Use a computer algebra system to find $P_{10}(x)$ and $Q_{10}(x)$, the Taylor polynomials of degree 10 about $x = 0$ for $\sin^2 x$ and $\cos^2 x$.

(b)
What similarities do you observe between the two polynomials? Explain your observation in terms of properties of sine and cosine.

61.
(a)
Use your computer algebra system to find $P_7(x)$ and $Q_7(x)$, the Taylor polynomials of degree 7 about $x = 0$ for $f(x) = \sin x$ and $g(x) = \sin x \cos x$.

ANSWER

WORKED SOLUTION

(b)
Find the ratio between the coefficient of x^3 in the two polynomials. Do the same for the coefficients of x^5 and x^7.

ANSWER

WORKED SOLUTION

(c)
Describe the pattern in the ratios that you computed in part b. Explain it using the identity $\sin(2x) = 2 \sin x \cos x$.

WORKED SOLUTION

62.
(a)
Calculate the equation of the tangent line to the function $f(x) = x^2$ at $x = 2$. Do the same calculation for $g(x) = x^3 - 4x^2 + 8x - 7$ at $x = 1$ and for $h(x) = 2x^3 + 4x^2 - 3x + 7$ at $x = -1$.

(b)
Use a computer algebra system to divide $f(x)$ by $(x - 2)^2$, giving your result in the form

$$\frac{f(x)}{(x-2)^2} = q(x) + \frac{r(x)}{(x-2)^2},$$

()
where $q(x)$ is the quotient and $r(x)$ is the remainder. In addition, divide $g(x)$ by $(x - 1)^2$ and $h(x)$ by $(x + 1)^2$.

(c)
For each of the functions, f, g, h, compare your answers to part a with the remainder, $r(x)$. What do you notice? Make a conjecture about the tangent line to a polynomial $p(x)$ at the point $x = a$ and the remainder, $r(x)$, obtained from dividing $p(x)$ by $(x - a)^2$.

(d)
Use the Taylor expansion of $p(x)$ about $x = a$ to prove your conjecture.[14] See "*Tangents Without*

Calculus" by Jorge Aarao, *The College Mathematics Journal* Vol 31, No 5, Nov 2000 (Mathematical Association of America).

63.
Let $f(x) = \frac{x}{e^x - 1} + \frac{x}{2}$. Although the formula for f is not defined at $x = 0$, we can make f continuous by setting $f(0) = 1$. If we do this, f has a Taylor series about $x = 0$.
(a)
Use a computer algebra system to find $P_{10}(x)$, the Taylor polynomial of degree 10 about $x = 0$ for f.
ANSWER

(b)
What do you notice about the degrees of the terms in the polynomial? What property of f does this suggest?
ANSWER

(c)
Prove that f has the property suggested by part b.
ANSWER

64.
Let $S(x) = \int_0^x \sin(t^2)\ dt$.
(a)
Use a computer algebra system to find $P_{11}(x)$, the Taylor polynomial of degree 11 about $x = 0$, for $S(x)$.

(b)
What is the percentage error in the approximation of $S(1)$ by $P_{11}(1)$? What about the approximation of $S(2)$ by $P_{11}(2)$?