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REVIEW EXERCISES AND PROBLEMS FOR CHAPTER 8

EXERCISES

1.

Imagine a hard-boiled egg lying on its side cut into thin slices. First think about vertical slices and then horizontal ones. What would these slices look like? Sketch them.

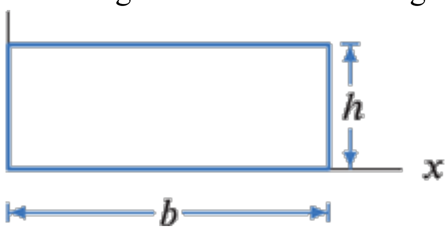
ANSWER

WORKED SOLUTION

For each region in Exercises 2-4, write a definite integral which represents its area. Evaluate the integral to derive a formula for the area.

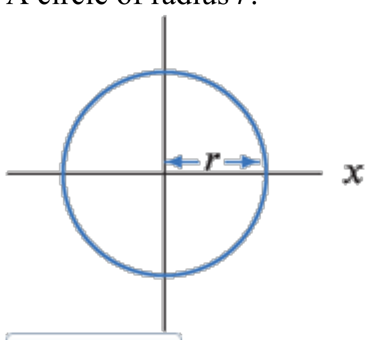
2.

A rectangle with base b and height h:



3.

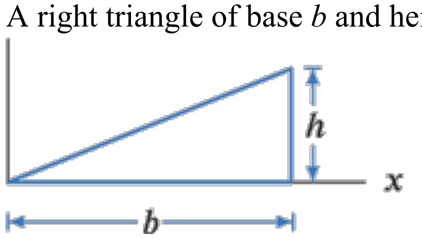
A circle of radius r:



ANSWER

4.

A right triangle of base b and height h:



5.

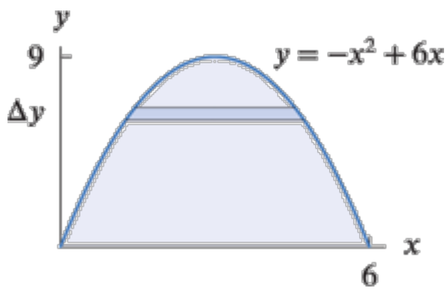


Figure 8.112

(a)

Write a Riemann sum approximating the area of the region in Figure 8.112, using horizontal strips as shown.

ANSWER

WORKED SOLUTION

(b)

Evaluate the corresponding integral.

ANSWER

WORKED SOLUTION

In Exercises 6-10, the region is rotated about the x-axis. Find the volume

6.

Bounded by $y = x^2 + 1$, the x-axis, $x = 0$, $x = 4$.

7.

Bounded by $y = \sqrt{x}$, x-axis, $x = 1$, $x = 2$.

ANSWER

8.

Bounded by $y = e^{-2x}$, the x-axis, $x = 0$, $x = 1$.

9.

Bounded by $y = 4 - x^2$ and the x-axis.

ANSWER

WORKED SOLUTION

10.

Bounded by $y = 2x$, $y = x$, $x = 0$, $x = 3$.

11.

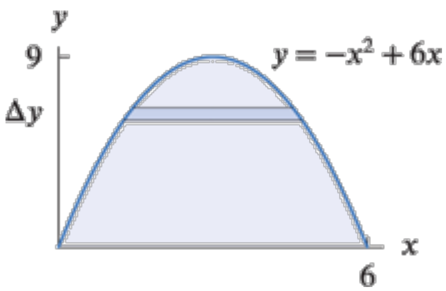


Figure 8.113

(a)

The region in Figure 8.113 is rotated around the y-axis. Using the strip shown, write an integral giving the volume.

ANSWER

(b)

Evaluate the integral.

ANSWER

Exercises 12-17 refer to the regions marked in Figure 8.114. Set up, but do not evaluate, an integral that represents the volume obtained when the region is rotated about the given axis.

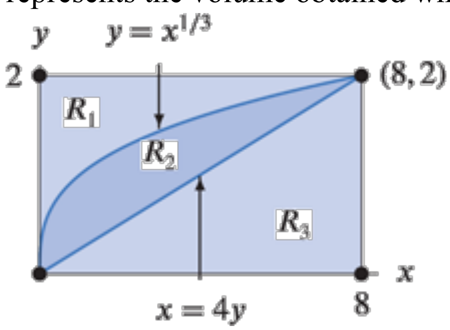


Figure 8.114

12.

R_2 about the x-axis

13.

R_1 about the y-axis

ANSWER

WORKED SOLUTION

14.

R_1 about the line $y = -2$

15.

R_3 about the line $x = 10$

ANSWER

16.

R_3 about the line $y = 3$

17.

R_2 about the line $x = -3$

ANSWER

WORKED SOLUTION

18.

Find the volume of the region in Figure 8.115, given that the radius, r of the circular slice at h is $r = \sqrt{h}$.

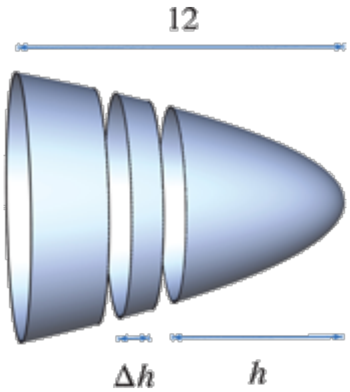


Figure 8.115

19.

Find, by slicing, the volume of a cone whose height is 3 cm and whose base radius is 1 cm. Slice the cone as shown in Figure 8.6.

ANSWER

For the curves described in Exercises 20-21, write the integral that gives the exact length of the curve; do not evaluate it.

20.

One arch of the sine curve, from $x = 0$ to $x = \pi$.

21.

The ellipse with equation $(x^2/a^2) + (y^2/b^2) = 1$.

ANSWER

WORKED SOLUTION

In Exercises 22-23, find the arc length of the function from $x = 0$ to $x = 3$. Use a graph to explain why your answer is reasonable.

22.

$f(x) = \sin x$

23.

$f(x) = 5x^2$

ANSWER

For Exercises 24-26, find the arc lengths.

24.

$$f(x) = \sqrt{1 - x^2} \text{ from } x = 0 \text{ to } x = 1$$

25.
$f(x) = e^x$ from $x = 1$ to $x = 2$

ANSWER

WORKED SOLUTION

26.
$f(x) = \frac{1}{3}x^3 + \frac{1}{4x}$ from $x = 1$ to $x = 2$.

■In Exercises 27-28, find the length of the parametric curves. Give exact answers if possible.

27.
$x = 3 \cos t$, $y = 2 \sin t$, for $0 \le t \le 2\pi$.

ANSWER

28.
$x = 1 + \cos(2t)$, $y = 3 + \sin(2t)$, for $0 \le t \le \pi$.

■In Exercises 29-33, let $f(x) = x^p$, for $x \ge 0$ and $p > 1$. Note that $f(0) = 0$, $f(1) = 1$, and f is increasing with a concaveup graph. Use geometrical arguments to order the given quantities.

29.

$$\int_0^1 f(x)\,dx \text{ and } \frac{1}{2}$$

ANSWER

WORKED SOLUTION

30.

$$\int_0^{0.5} f'(x)\,dx \text{ and } \frac{1}{2}$$

31.

$$\int_0^1 f^{-1}(x)\,dx \text{ and } \frac{1}{2}$$

ANSWER

32.

$$\int_0^1 \pi(f(x))^2\,dx \text{ and } \frac{\pi}{3}$$

33.

$$\int_0^1 \sqrt{1+\left(f'(x)\right)^2}\,dx \text{ and } \sqrt{2}$$

ANSWER ⊕

WORKED SOLUTION ⊕

PROBLEMS

34.

(a)

Find the area of the region between $y = x^2$ and $y = 2x$.

(b)

Find the volume of the solid of revolution if this region is rotated about the x-axis.

(c)

Find the length of the perimeter of this region.

35.

The integral $\int_0^2 \left(\sqrt{4-x^2} - \left(-\sqrt{4-x^2}\right)\right) dx$ represents the area of a region in the plane. Sketch this region.

ANSWER ⊕

In Problems 36-37, set up definite integral(s) to find the volume obtained when the region between $y = x^2$ and $y = 5x$ is rotated about the given axis. Do not evaluate the integral(s).

36.

The line $y = 30$

37.

The line $x = 8$

ANSWER ⊕

WORKED SOLUTION ⊕

38.

(a)

Sketch the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $x = 1$, and $y = 0$ around the line $y = 0$.

(b)

Approximate its volume by Riemann sums, showing the volume represented by each term in your sum on the sketch.

(c)

Now find the volume of this solid using an integral.

39.
Using the region of Problem 38, find the volume when it is rotated around
(a)
The line $y = 1$.
ANSWER

(b)
The y-axis.
ANSWER

For Problems 40-42, set up and compute an integral giving the volume of the solid of revolution.
40.
Bounded by $y = \sin x$, $y = 0.5x$, $x = 0$, $x = 1.9$;
(a)
Rotated about the x-axis.

(b)
Rotated about $y = 5$.

41.
Bounded by $y = 2x$, the x-axis, $x = 0$, $x = 4$. Axis: $y = -5$.
ANSWER
WORKED SOLUTION

42.
Bounded by $y = x^2$, the x-axis, $x = 0$, $x = 3$;
(a)
Rotated about $y = -2$.

(b)
Rotated about $y = 10$.

Problems 43-48 concern the region bounded by the quarter circle $x^2 + y^2 = 1$, with $x \geq 0$, $y \geq 0$. Find the volume of the following solids.
43.
The solid obtained by rotating the region about the x-axis.
ANSWER

44.
The solid obtained by rotating the region about the line $x = -2$.

45.
The solid obtained by rotating the region about the line $x = 1$.
ANSWER

WORKED SOLUTION ⊕

46.
The solid whose base is the region and whose cross-sections perpendicular to the x-axis are squares.

47.
The solid whose base is the region and whose cross-sections perpendicular to the y-axis are semicircles.
ANSWER ⊕

48.
The solid whose base is the region and whose cross-section perpendicular to the y-axis is an isosceles right triangle with one leg in the region.

■ In Problems 49-50, what does the expression represent geometrically in terms of the function $f(x) = x(x - 3)^2$? Do not evaluate the expressions.
49.

$$\int_0^3 x(x-3)^4\,dx$$

ANSWER ⊕
WORKED SOLUTION ⊕

50.

$$\int_0^3 \pi x^2(x-3)^4\,dx$$

51.
The catenary $\cosh x = \frac{1}{2}(e^x + e^{-x})$ represents the shape of a hanging cable. Find the exact length of this catenary between $x = -1$ and $x = 1$.
ANSWER ⊕

52.
The reflector behind a car headlight is made in the shape of the parabola $x = \frac{4}{9}y^2$, with a circular cross-section, as shown in Figure 8.116.

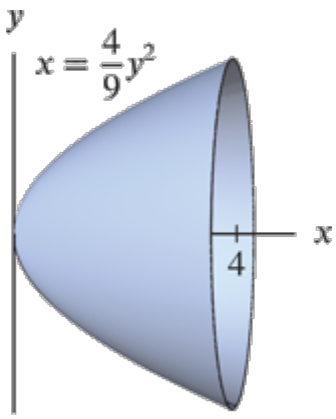


Figure 8.116

(a)

Find a Riemann sum approximating the volume contained by this headlight.

(b)

Find the volume exactly.

53.

In this problem, you will derive the formula for the volume of a right circular cone with height l and base radius b by rotating the line $y = ax$ from $x = 0$ to $x = l$ around the x-axis. See Figure 8.117.

(a)

What value should you choose for a such that the cone will have height l and base radius b?

ANSWER ⊕

WORKED SOLUTION ⊕

(b)

Given this value of a, find the volume of the cone.

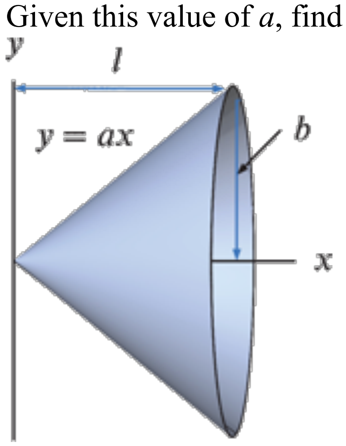


Figure 8.117

ANSWER ⊕

WORKED SOLUTION ⊕

54.

Figure 8.118 shows a cross section through an apple. (Scale: One division = 1/2 inch.)

(a)

Give a rough estimate for the volume of this apple (in cubic inches).

(b)

The density of these apples is about 0.03 lb/in^3 (a little less than the density of water—as you might expect, since apples float). Estimate how much this apple would cost. (They go for 80 cents a pound.)

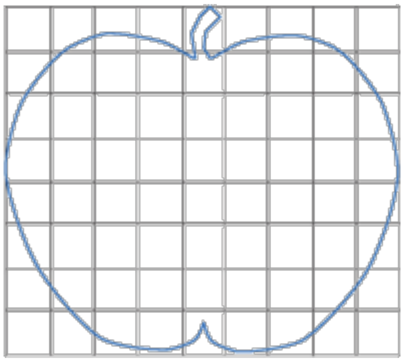


Figure 8.118

55.

The circle $x^2 + y^2 = 1$ is rotated about the line $y = 3$ forming a torus (a doughnut-shaped figure). Find the volume of this torus.

ANSWER

56.

A 100 cm long gutter is made of three strips of metal, each 5 cm wide; Figure 8.119 shows a cross-section.

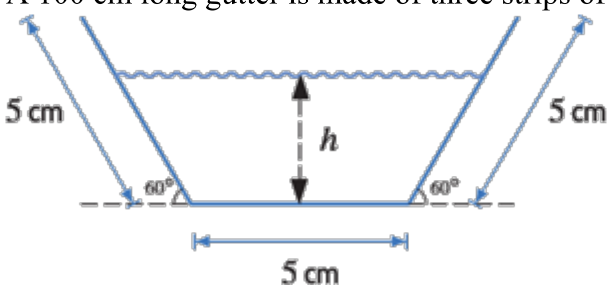


Figure 8.119

(a)

Find the volume of water in the gutter when the depth is h cm.

(b)

What is the maximum value of h?

(c)

What is the maximum volume ofwater that the gutter can hold?

(d)

If the gutter is filled with half the maximum volume of water, is the depth larger or smaller than half of the answer to part b? Explain how you can answer without any calculation.

(e)

Find the depth of the water when the gutter contains half the maximum possible volume.

57.

Find a curve whose arc length is $\int_3^8 \sqrt{1 + e^{6t}}\, dt$.

ANSWER ⊕

WORKED SOLUTION ⊕

58.

Water is flowing in a cylindrical pipe of radius 1 inch. Because water is viscous and sticks to the pipe, the rate of flow varies with the distance from the center. The speed of the water at a distance r inches from the center is $10(1 - r^2)$ inches per second. What is the rate (in cubic inches per second) at which water is flowing through the pipe?

Problems 59-63 concern C, the circle $r = 2\,a\cos\theta$, for $-\pi/2 \le \theta \le \pi/2$, of radius $a > 0$ centered at the point $(x, y) = (a, 0)$ on the x-axis.

59.

By converting to Cartesian coordinates, show that $r = 2a\cos\theta$ gives the circle described.

ANSWER ⊕

60.

Find the area of the circle C by integrating in polar coordinates.

61.

Find the area of the region enclosed by C and outside the circle of radius a centered at the origin. What percent is this of the area of C?

ANSWER ⊕

WORKED SOLUTION ⊕

62.

(a)

Find the slope of C at the angle θ.

(b)

At what value of θ does the maximum y-value occur?

63.

Calculate the arc length of C using polar coordinates.

ANSWER ⊕

64.

Write a definite integral for the volume of the bounded region formed by rotating the graph of $y = (x - 1)^2(x + 2)$ around the x-axis. You need not evaluate this integral.

65.

Find the center of mass of a system containing four identical point masses of 3 gm, located at $x = -5, -3, 2, 7$.

ANSWER ⊕

WORKED SOLUTION ⊕

66.

A metal plate, with constant density 2 gm/cm^2, has a shape bounded by the two curves $y = x^2$ and $y = \sqrt{x}$, with $0 \leq x \leq 1$, and x, y in cm.

(a)

Find the total mass of the plate.

(b)

Because of the symmetry of the plate about the line $y = x$, we have $\bar{x} = \bar{y}$. Sketch the plate and decide, on the basis of the shape, whether $\bar{x}$ is less than or greater than 1/2.

(c)

Find $\bar{x}$ and $\bar{y}$.

67.

A 200-lb weight is attached to a 20-foot rope and dangling from the roof of a building. The rope weighs 2 lb/ft. Find the work done in lifting the weight to the roof.

ANSWER ⊕

68.

A 10 ft pole weighing 20 lbs lies flat on the ground. Keeping one end of the pole braced on the ground, the other end is lifted until the pole stands vertically. Once the pole is upright, the segment of length Δx at height x has been raised a vertical distance of x ft. How much work is done to raise the pole vertically?

69.

Water is raised from a well 40 ft deep by a bucket attached to a rope. When the bucket is full, it weighs 30 lb. However, a leak in the bucket causes it to lose water at a rate of 1/4 lb for each foot that the bucket is raised. Neglecting the weight of the rope, find the work done in raising the bucket to the top.

ANSWER ⊕

WORKED SOLUTION ⊕

70.

A rectangular water tank has length 20 ft, width 10 ft, and depth 15 ft. If the tank is full, how much work does it take to pump all the water out?

71.

A fuel oil tank is an upright cylinder, buried so that its circular top is 10 feet beneath ground level. The tank has a radius of 5 feet and is 15 feet high, although the current oil level is only 6 feet deep. Calculate the work required to pump all of the oil to the surface. Oil weighs 50 lb/ft^3.

ANSWER ⊕

72.

An underground tank filled with gasoline of density 42 lb/ft^3 is a hemisphere of radius 5 ft, as in Figure

8.120. Use an integral to find the work to pump the gasoline over the top of the tank.

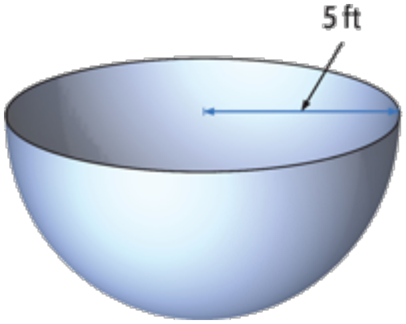


Figure 8.120

73.

(a)

A reservoir has a dam at one end. The dam is a rectangular wall, 1000 feet long and 50 feet high. Approximate the total force of the water on the dam by a Riemann sum.

ANSWER ⊕

WORKED SOLUTION ⊕

(b)

Write an integral which represents the force, and evaluate it.

ANSWER ⊕

WORKED SOLUTION ⊕

74.

A crane lifts a 1000 lb object to a height of 20 ft using chain that weighs 2 lb/ft. If the crane arm is at a height of 50 ft, find the work required.

75.

A cylindrical barrel, standing upright on its circular end, contains muddy water. The top of the barrel, which has diameter 1 meter, is open. The height of the barrel is 1.8 meter and it is filled to a depth of 1.5 meter. The density of the water at a depth of h meters below the surface is given by $\delta(h) = 1 + kh$ kg/m^3, where k is a positive constant. Find the total work done to pump the muddy water to the top rim of the barrel. (You can leave π, k, and g in your answer.)

ANSWER ⊕

76.

Find the present and future values of an income stream of \$3000 per year over a 15-year period, assuming a 6% annual interest rate compounded continuously.

77.

In 1980, before the unification of Germany in 1990 and the introduction of the Euro, West Germany made a loan of 20 billion Deutsche Marks to the Soviet Union, to be used for the construction of a natural gas pipeline connecting Siberia to Western Russia, and continuing to West Germany (Urengoi-Uschgorod-Berlin). Assume that the deal was as follows: In 1985, upon completion of the pipeline, the Soviet Union would deliver natural gas to West Germany, at a constant rate, for all future times. Assuming a constant price of natural gas of 0.10 Deutsche Mark per cubic meter, and assuming West Germany expects 10% annual interest on its investment (compounded continuously), at what rate does the Soviet Union have to deliver the gas, in billions of cubic meters per year? Keep in mind that delivery of gas could not begin until the pipeline was completed. Thus, West Germany received no return on its investment until after five years had passed. (Note: A more complex deal of this type was actually made between the two countries.)

ANSWER ⊕

WORKED SOLUTION ⊕

78.

A nuclear power plant produces strontium-90 at a rate of 3 kg/yr. How much of the strontium produced since 1971 (when the plant opened) was still around in 1992? (The half-life of strontium-90 is 28 years.)

79.

Mt. Shasta is a cone-like volcano whose radius at an elevation of h feet above sea level is approximately $(3.5 \cdot 10^5)/\sqrt{h+600}$ feet. Its bottom is 400 feet above sea level, and its top is 14,400 feet above sea level. See Figure 8.121. (Note: Mt. Shasta is in northern California, and for some time was thought to be the highest point in the US outside Alaska.)

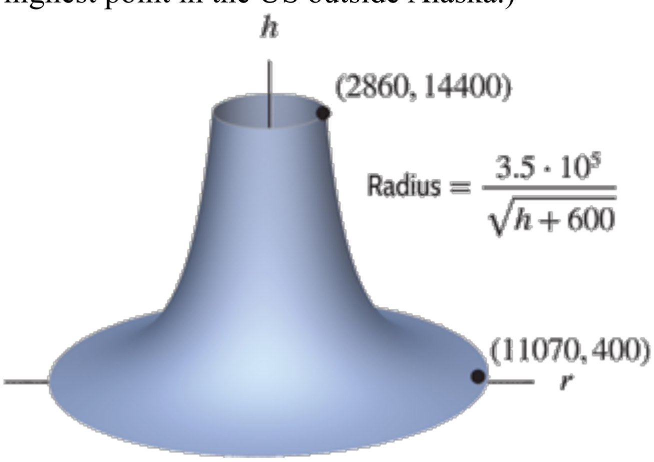


Figure 8.121: Mt. Shasta

(a)

Give a Riemann sum approximating the volume of Mt. Shasta.

ANSWER ⊕

(b)

Find the volume in cubic feet.

ANSWER ⊕

80.

Figure 8.122 shows an ancient Greek water clock called a clepsydra, which is designed so that the depth of the water decreases at a constant rate as the water runs out a hole in the bottom. This design allows the hours to be marked by a uniform scale. The tank of the clepsydra is a volume of revolution about a vertical axis. According to Torricelli's law, the exit speed of the water flowing through the hole is proportional to the square root of the depth of the water. Use this to find the formula $y = f(x)$ for this profile, assuming that $f(1)$

= 1.

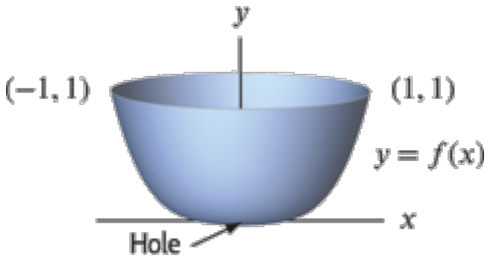


Figure 8.122

81.

Suppose that $P(t)$ is the cumulative distribution function for age in the US, where x is measured in years. What is the meaning of the statement $P(70) = 0.92$?

ANSWER

WORKED SOLUTION

82.

Figure 8.123 shows the distribution of the velocity of molecules in two gases. In which gas is the average velocity larger?

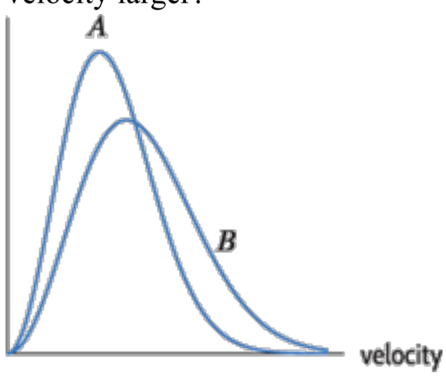


Figure 8.123

83.

A radiation detector is a circular disk which registers photons hitting it. The probability that a photon hitting the disk at a distance r from the center is actually detected is given by $S(r)$. A radiation detector of radius R is bombarded by constant radiation of N photons per second per unit area. Write an integral representing the number of photons per second registered by the detector.

ANSWER

84.

Housing prices depend on the distance in miles r from a city center according to

$$p(r) = 400e^{-0.2r^2}, \quad \text{price in \$1000s.}$$

()

Assuming 1000 houses per square mile, what is the total value of the houses within 7 miles of the city center?

85.

A blood vessel is cylindrical with radius R and length l. The blood near the boundary moves slowly; blood at the center moves the fastest. The velocity, v, of the blood at a distance r from the center of the artery is given by

$$v = \frac{P}{4\eta l}(R^2 - r^2)$$

()

where P is the pressure difference between the ends of the blood vessel and η is the viscosity of blood.

(a)

Find the rate at which the blood is flowing down the blood vessel. (Give your answer as a volume per unit time.)

ANSWER ⊕

WORKED SOLUTION ⊕

(b)

Show that your result agrees with Poiseuille's Law, which says that the rate at which blood is flowing down the blood vessel is proportional to the radius of the blood vessel to the fourth power.

WORKED SOLUTION ⊕

86.

A car moving at a speed of v mph achieves $25 + 0.1v$ mpg (miles per gallon) for v between 20 and 60 mph. Your speed as a function of time, t, in hours, is given by

$$v = 50\frac{t}{t+1}\text{mph.}$$

()

How many gallons of gas do you consume between $t = 2$ and $t = 3$?

87.

A bowl is made by rotating the curve $y = ax^2$ around the y-axis (a is a positive constant).

(a)

The bowl is filled with water to depth h. What is the volume of water in the bowl? (Your answer will contain a and h.)

ANSWER ⊕

(b)

What is the area of the surface of the water if the bowl is filled to depth h? (Your answer will contain a and h.)

ANSWER ⊕

(c)

Water is evaporating from the surface of the bowl at a rate proportional to the surface area, with proportionality constant k. Find a differential equation satisfied by h as a function of time, t. (That is, find an equation for dh/dt.)

ANSWER ⊕

(d)

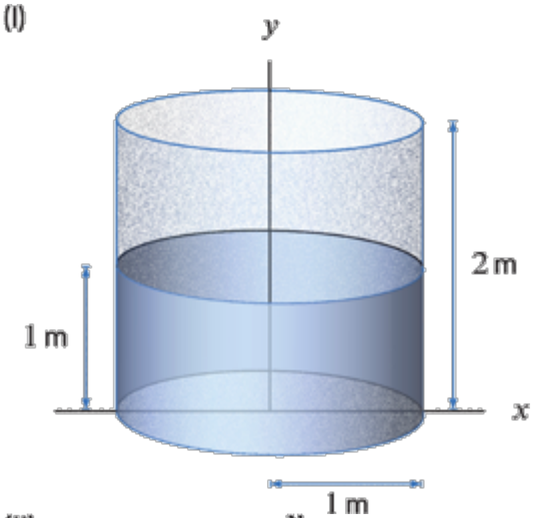
If the water starts at depth h_0, find the time taken for all the water to evaporate.

ANSWER ⊕

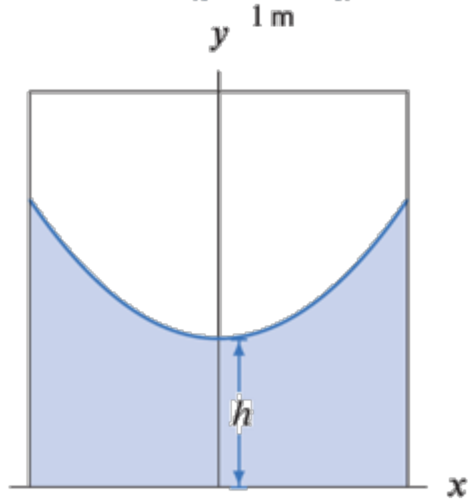
88.

A cylindrical centrifuge of radius 1 m and height 2 m is filled with water to a depth of 1 meter (see Figure 8.124(I)). As the centrifuge accelerates, the water level rises along the wall and drops in the center; the cross-section will be a parabola. (See Figure 8.124(II).)

(I)



(II)



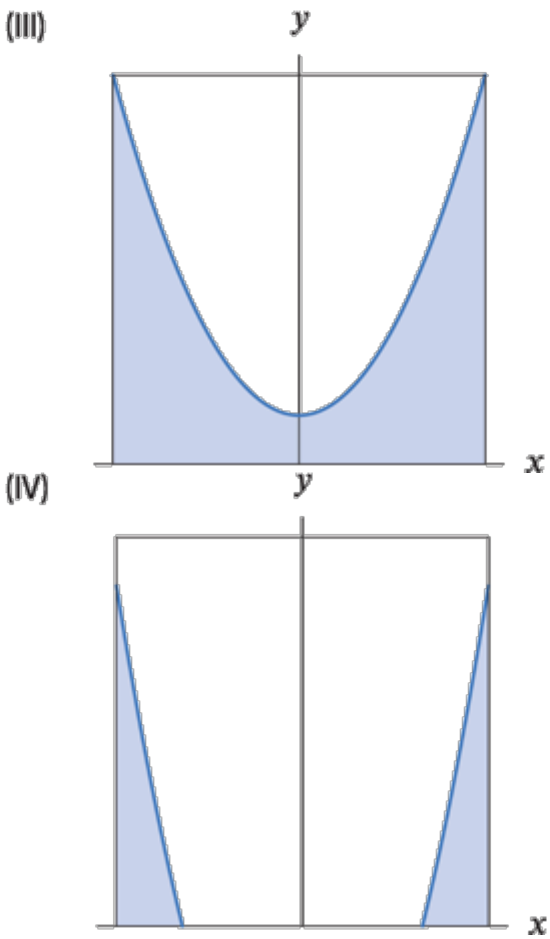


Figure 8.124

(a)

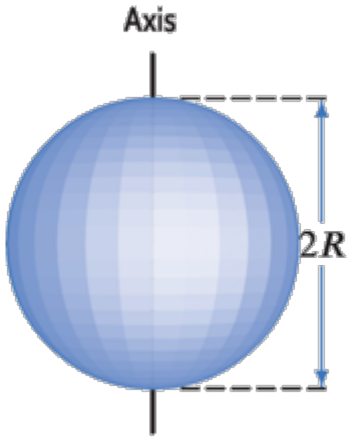
Find the equation of the parabola in Figure 8.124(II) in terms of h, the depth of the water at its lowest point.

(b)

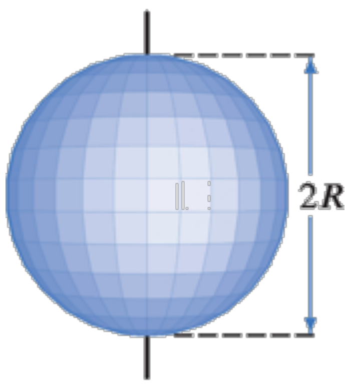
As the centrifuge rotates faster and faster, either water will be spilled out the top, as in Figure 8.124(III), or the bottom of the centrifuge will be exposed, as in Figure 8.124(IV). Which happens first?

In Problems 89-90, you are given two objects that have the same mass M, the same radius R, and the same angular velocity about the indicated axes (say, one revolution per minute). Use reasoning (not computation) to determine which of the two objects has the greater kinetic energy. (The kinetic energy of a particle of mass m with speed v is $\frac{1}{2}mv^2$.)

89.



Solid sphere about any diameter

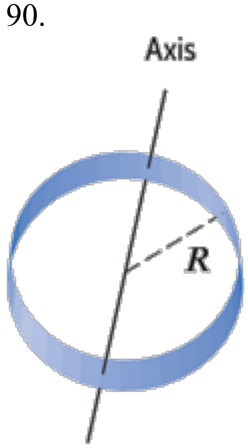


Thin spherical shell about any diameter

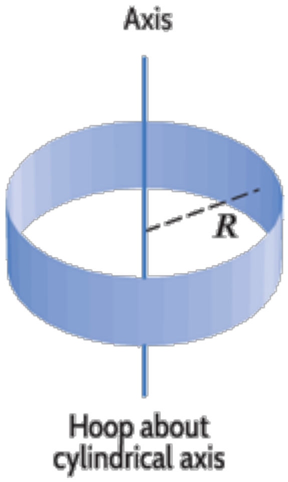
ANSWER ⊕

WORKED SOLUTION ⊕

90.



Hoop about any diameter



CAS Challenge Problems

91.

For a positive constant a, consider the curve

$$y = \sqrt{\frac{x^3}{a-x}}, 0 \le x < a.$$

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(a)

Using a computer algebra system, show that for $0 \le t < \pi/2$, the point with coordinates (x, y) lies on the curve if:

$$x = a\ \sin^2 t, y = \frac{a\ \sin^3 t}{\cos\ t}.$$

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(b)

A solid is obtained by rotating the curve about its asymptote at $x = a$. Use horizontal slicing to write an integral in terms of x and y that represents the volume of this solid.

(c)

Use part a to substitute in the integral for both x and y in terms of t. Use a computer algebra system or trigonometric identities to calculate the volume of the solid.

ANSWER

For Problems 92-93, define $A(t)$ to be the arc length of the graph of $y = f(x)$ from $x = 0$ to $x = t$, for $t \ge 0$.

- (a)Use the integral expression for arc length and a computer algebra system to obtain a formula for $A(t)$.
- (b)Graph $A(t)$ for $0 \le t \le 10$. What simple function does $A(t)$ look like? What does this tell you about the approximate value of $A(t)$ for large t?
- (c)In order to estimate arc length visually, you need the same scales on both axes, so that the lengths are

not distorted in one direction. Draw a graph of $f(x)$ with viewing window $0 \le x \le 100$, $0 \le y \le 100$. Explain what you noticed in part (b) in terms of this graph.

92.
$f(x) = x^2$

93.

$$f(x) = \sqrt{x}$$

ANSWER ⊕
WORKED SOLUTION ⊕

94.
A bead is formed by drilling a cylindrical hole of circular cross section and radius a through a sphere of radius $r > a$, the axis of the hole passing through the center of the sphere.
(a)
Write a definite integral expressing the volume of the bead.

(b)
Find a formula for the bead by evaluating the definite integral in part a.