Print this page

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER 9

EXERCISES

In Exercises 1-8, find the sum of the series.

1.

$$3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\cdots+\frac{3}{2^{10}}$$

ANSWER

WORKED SOLUTION

2.

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots$$

3.

$125+100+80+\cdots+125(0.8)^{20}$

ANSWER

4.

$(0.5)^3+(0.5)^4+\cdots+(0.5)^k$

5.

$b^5+b^6+b^7+b^8+b^9+b^{10}$

ANSWER

WORKED SOLUTION

6.

$$\sum_{n=4}^{\infty}\left(\frac{1}{3}\right)^n$$

7.

$$\sum_{n=4}^{20}\left(\frac{1}{3}\right)^n$$

ANSWER

8.

$$\sum_{n=0}^{\infty} \frac{3^n + 5}{4^n}$$

In Exercises 9-12, find the the first four partial sums of the geometric series, a formula for the n^{th} partial sum, and the sum of the series, if it exists.

9.

$$36 + 12 + 4 + \frac{4}{3} + \frac{4}{9} + \cdots$$

ANSWER

WORKED SOLUTION

10.
$1280 - 960 + 720 - 540 + 405 - \cdots$

11.
$-810 + 540 - 360 + 240 - 160 + \cdots$

ANSWER

12.
$2 + 6z + 18z^2 + 54z^3 + \cdots$

In Exercises 13-16, does the sequence converge or diverge? If a sequence converges, find its limit.

13.

$$\frac{3 + 4n}{5 + 7n}$$

ANSWER

WORKED SOLUTION

14.

$$(-1)^n \frac{(n+1)}{n}$$

15.

$$\sin\left(\frac{\pi}{4} n\right)$$

ANSWER

16.

$$\frac{2^n}{n^3}$$

In Exercises 17-20, use the integral test to decide whether the series converges or diverges.

17.

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

ANSWER

WORKED SOLUTION

18.

$$\sum_{n=1}^{\infty} \frac{3n^2 + 2n}{n^3 + n^2 + 1}$$

19.

$$\sum_{n=0}^{\infty} ne^{-n^2}$$

ANSWER

20.

$$\sum_{n=2}^{\infty} \frac{2}{n^2 - 1}$$

In Exercises 21-23, use the ratio test to decide if the series converges or diverges.

21.

$$\sum_{n=1}^{\infty} \frac{1}{2^n n!}$$

ANSWER

WORKED SOLUTION

22.

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{5^n}$$

23.

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!}$$

ANSWER

In Exercises 24-25, use the alternating series test to decide whether the series converges.

24.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$$

25.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^2+1}}$$

ANSWER

WORKED SOLUTION

In Exercises 26-29, determine whether the series is absolutely convergent, conditionally convergent, or divergent.

26.

$$\sum \frac{(-1)^n}{n^{1/2}}$$

27.

$$\sum (-1)^n \left(1 + \frac{1}{n^2}\right)$$

ANSWER

28.

$$\sum \frac{(-1)^{n-1} \ln n}{n}$$

29.

$$\sum \frac{(-1)^{n-1}}{\arctan n}$$

ANSWER

WORKED SOLUTION

In Exercises 30-31, use the comparison test to confirm the statement.

30.

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges, so $\sum_{n=1}^{\infty} \left(\frac{n^2}{3n^2+4}\right)^n$ converges.

31.

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\sum_{n=1}^{\infty} \frac{1}{n\sin^2 n}$ diverges.

In Exercises 32-35, use the limit comparison test to determine whether the series converges or diverges.

32.

$$\sum \frac{\sqrt{n-1}}{n^2+3}$$

33.

$$\sum \frac{n^3-2n^2+n+1}{n^5-2}$$

ANSWER

WORKED SOLUTION

34.

$$\sum \sin \frac{1}{n^2}$$

35.

$$\sum \frac{1}{\sqrt{n^3-1}}$$

ANSWER

In Exercises 36-57, determine whether the series converges.

36.

$$\sum_{n=1}^{\infty} \frac{1}{n+1}$$

37.

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

ANSWER ⊕

WORKED SOLUTION ⊕

38.

$$\sum_{n=3}^{\infty} \frac{2}{\sqrt{n-2}}$$

39.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}+1}$$

ANSWER ⊕

40.

$$\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$$

41.

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$$

ANSWER ⊕

WORKED SOLUTION ⊕

42.

$$\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$$

43.

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$$

ANSWER

44.

$$\sum_{n=1}^{\infty} \frac{n^2 + 2^n}{n^2 2^n}$$

45.

$$\sum_{n=1}^{\infty} \frac{3^{2n}}{(2n)!}$$

ANSWER

WORKED SOLUTION

46.

$$\sum_{n=1}^{\infty} 2^{-n} \frac{(n+1)}{(n+2)}$$

47.

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{(2n+1)!}$$

ANSWER

48.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{\sqrt{n}}$$

49.

$$\sum_{n=0}^{\infty} \frac{2 + 3^n}{5^n}$$

ANSWER

WORKED SOLUTION

50.

$$\sum_{n=1}^{\infty} \left(\frac{1+5n}{4n}\right)^n$$

51.

$$\sum_{n=1}^{\infty} \frac{1}{2+\sin n}$$

ANSWER

52.

$$\sum_{n=3}^{\infty} \frac{1}{(2n-5)^3}$$

53.

$$\sum_{n=2}^{\infty} \frac{1}{n^3-3}$$

ANSWER

WORKED SOLUTION

54.

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^3}$$

55.

$$\sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k}\right)$$

ANSWER

56.

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

57.

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$$

ANSWER ⊕

WORKED SOLUTION ⊕

In Exercises 58-61, find the radius of convergence.

58.

$$\sum_{n=1}^{\infty} \frac{(2n)!x^n}{(n!)^2}$$

59.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!+1}$$

ANSWER ⊕

60.

$x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + \cdots$

61.

$$\frac{x}{3} + \frac{2x^2}{5} + \frac{3x^3}{7} + \frac{4x^4}{9} + \frac{5x^5}{11} + \cdots$$

ANSWER ⊕

WORKED SOLUTION ⊕

In Exercises 62-65, find the interval of convergence.

62.

$$\sum_{n=1}^{\infty} \frac{x^n}{3^n n^2}$$

63.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(x-2)^n}{5^n}$$

ANSWER ⊕

64.

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

65.

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

ANSWER ⊕

WORKED SOLUTION ⊕

PROBLEMS

66.

Write the first four terms of the sequence given by

$$s_n = \frac{(-1)^n (2n+1)^2}{2^{2n-1} + (-1)^{n+1}}, \quad n \geq 1.$$

()

In Problems 67-68, find a possible formula for the general term of the sequence.

67.

$s_1, s_2, s_3, s_4, s_5, \ldots = 5, 7, 9, 11, 13, \ldots$

ANSWER ⊕

68.

$t_1, t_2, t_3, t_4, t_5, \ldots = 9, 25, 49, 81, 121, \ldots$

In Problems 69-70, let $a_1 = 5$, $b_1 = 10$ and, for $n > 1$,

$$a_n = a_{n-1} + 2n \quad \text{and} \quad b_n = b_{n-1} + a_{n-1}.$$

()

69.

Give the values of a_2, a_3, a_4.

ANSWER ⊕

WORKED SOLUTION ⊕

70.

Give the values of b_2, b_3, b_4, b_5.

71.

For $r > 0$, how does the convergence of the following series depend on r?

$$\sum_{n=1}^{\infty} \frac{n^r + r^n}{n^r r^n}$$

()

ANSWER

72.

The series $\sum C_n(x-2)^n$ converges when $x = 4$ and diverges when $x = 6$. Decide whether each of the following statements is true or false, or whether this cannot be determined.

(a)

The power series converges when $x = 7$.

(b)

The power series diverges when $x = 1$.

(c)

The power series converges when $x = 0.5$.

(d)

The power series diverges when $x = 5$.

(e)

The power series converges when $x = -3$.

73.

For all the t-values for which it converges, the function h is defined by the series

$$h(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}.$$

()

(a)

What is the domain of h?

ANSWER

WORKED SOLUTION

(b)

Is h odd, even, or neither?

ANSWER

WORKED SOLUTION

(c)

Assuming that derivatives can be computed term by term, show that

$$h''(t) = -h(t).$$

()

WORKED SOLUTION

74.

A \$200,000 loan is to be repaid over 20 years in equal monthly installments of $\$M$, beginning at the end of the first month. Find the monthly payment if the loan is at an annual rate of 9%, compounded monthly. [Hint: Find an expression for the present value of the sum of all of the monthly payments, set it equal to \$200,000, and solve for M.]

75.

The extraction rate of a mineral is currently 12 million tons a year, but this rate is expected to fall by 5% each year. What minimum level of world reserves would allow extraction to continue indefinitely?

ANSWER

76.

A new car costs \$30,000; it loses 10% of its value each year. Maintenance is \$500 the first year and increases by 20% annually.

(a)

Find a formula for l_n, the value lost by the car in year n.

(b)

Find a formula for m_n, the maintenance expenses in year n.

(c)

In what year do maintenance expenses first exceed the value lost by the car?

Problems 77-79 are about *bonds*, which are issued by a government to raise money. An individual who buys a \$1000 bond gives the government \$1000 and in return receives a fixed sum of money, called the *coupon*, every six months or every year for the life of the bond. At the time of the last coupon, the individual also gets back the \$1000, or principal.

77.

What is the present value of a \$1000 bond which pays \$50 a year for 10 years, starting one year from now? Assume the interest rate is 6% per year, compounded annually.

ANSWER

WORKED SOLUTION

78.

What is the present value of a \$1000 bond which pays \$50 a year for 10 years, starting one year from now? Assume the interest rate is 4% per year, compounded annually.

79.

In the nineteenth century, the railroads issued 100-year bonds. Consider a \$100 bond which paid \$5 a year, starting a year after it was sold. Assume interest rates are 4% per year, compounded annually.

(a)

Find the present value of the bond.

ANSWER ⊕

(b)
Suppose that instead of maturing in 100 years, the bond was to have paid \$5 a year forever. This time the principal, \$100, is never repaid. What is the present value of the bond?

ANSWER ⊕

80.
Cephalexin is an antibiotic with a half-life in the body of 0.9 hours, taken in tablets of 250 mg every six hours.

(a)
What percentage of the cephalexin in the body at the start of a six-hour period is still there at the end (assuming no tablets are taken during that time)?

(b)
Write an expression for Q_1, Q_2, Q_3, Q_4, where Q_n mg is the amount of cephalexin in the body right after the n^{th} tablet is taken.

(c)
Express Q_3, Q_4 in closed form and evaluate them.

(d)
Write an expression for Q_n and put it in closed form.

(e)
If the patient keeps taking the tablets, use your answer to part d to find the quantity of cephalexin in the body in the long run, right after taking a tablet.

81.
Before World War I, the British government issued what are called *consols*, which pay the owner or his heirs a fixed amount of money every year forever. (Cartoonists of the time described aristocrats living off such payments as “pickled in consols.”) What should a person expect to pay for consols which pay \$10 a year forever? Assume the first payment is one year from the date of purchase and that interest remains 4% per year, compounded annually. (\$ denotes pounds, the British unit of currency.)

ANSWER ⊕

WORKED SOLUTION ⊕

82.
This problem illustrates how banks create credit and can thereby lend out more money than has been deposited. Suppose that initially \$100 is deposited in a bank. Experience has shown bankers that on average only 8% of the money deposited is withdrawn by the owner at any time. Consequently, bankers feel free to lend out 92% of their deposits. Thus \$92 of the original \$100 is loaned out to other customers (to start a business, for example). This \$92 becomes someone else's income and, sooner or later, is redeposited in the bank. Thus 92% of \$92, or \$92(0.92) = \$84.64, is loaned out again and eventually redeposited. Of the \$84.64, the bank again loans out 92%, and so on.

(a)
Find the total amount of money deposited in the bank as a result of these transactions.

(b)
The total amount of money deposited divided by the original deposit is called the *credit multiplier*. Calculate the credit multiplier for this example and explain what this number tells us.

83.
Baby formula can contain bacteria which double in number every half hour at room temperature and every 10 hours in the refrigerator.[11] Iverson, C. and Forsythe, F., reported in "Baby Food Could Trigger Meningitis," www.newscientist.com, June 3, 2004. There are B_0 bacteria initially.
(a)
Write formulas for
(i)
R_n, the number of bacteria n hours later if the baby formula is kept at room temperature.
ANSWER ⊕

(ii)
F_n, the number of bacteria n hours later if the baby formula is kept in the refrigerator.
ANSWER ⊕

(iii)
Y_n, the ratio of the number of bacteria at room temperature to the number of bacteria in the refrigerator.
ANSWER ⊕

(b)
How many hours does it take before there are a million times as many bacteria in baby formula kept at room temperature as there are in baby formula kept in the refrigerator?
ANSWER ⊕

84.
The sequence 1, 5/8, 14/27, 15/32, … is defined by:

$$\begin{aligned} s_1 &= 1 \\ s_2 &= \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 + \left(\frac{2}{2}\right)^2\right) = \frac{5}{8} \\ s_3 &= \frac{1}{3}\left(\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{3}{3}\right)^2\right) = \frac{14}{27}. \end{aligned}$$

()
(a)
Extend the pattern and find s_5.

(b)
Write an expression for s_n using sigma notation.

(c)
Use Riemann sums to evaluate $\lim_{n\to\infty} s_n$.

85.

Estimate $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$ to within 0.01 of the actual sum of the series.

ANSWER

WORKED SOLUTION

86.

Is it possible to construct a convergent alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ for which $0 < a_{n+1} < a_n$ but $\lim_{n\to\infty} a_n \neq 0$?

87.
Suppose that $0 \le b_n \le 2^n$ for all n. Give two examples of series $\sum b_n$ that satisfy this condition, one that diverges and one that converges.

ANSWER

88.
Show that if $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ diverges.
[Hint: Assume that $\sum (a_n + b_n)$ converges and consider $\sum (a_n + b_n) - \sum a_n$.]

In Problems 89-93, the series $\sum a_n$ converges with $a_n > 0$ for all n. Does the series converge or diverge or is there not enough information to tell?

89.
$\sum a_n / n$

ANSWER

WORKED SOLUTION

90.
$\sum 1/a_n$

91.
$\sum n a_n$

ANSWER

92.
$\sum (a_n + a_n/2)$

93.

$$\sum a_n^2$$

ANSWER ⊕

WORKED SOLUTION ⊕

94.

Does $\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{1}{n}\right)$ converge or diverge? Does $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n}\right)$ converge or diverge? Is the statement "If $\sum a_n$ and $\sum b_n$ diverge, then $\sum(a_n + b_n)$ may or may not diverge" true?

95.

This problem shows how you can create a fractal called a *Cantor Set*. Take a line segment of length 1, divide it into three equal pieces and remove the middle piece. We are left with two smaller line segments. At the second stage, remove the middle third of each of the two segments. We now have four smaller line segments left. At the third stage, remove the middle third of each of the remaining segments. Continue in this manner.

(a)

Draw a picture that illustrates this process.

ANSWER ⊕

(b)

Find a series that gives the total length of the pieces we have removed after the n^{th} stage.

ANSWER ⊕

(c)

If we continue the process indefinitely, what is the total length of the pieces that we remove?

ANSWER ⊕

96.

Although the harmonic series does not converge, the partial sums grow very, very slowly. Take a right-hand sum approximating the integral of $f(x) = 1/x$ on the interval $[1, n]$, with $\Delta x = 1$, to show that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n} < \ln n.$$

()

If a computer could add a million terms of the harmonic series each second, estimate the sum after one year.

97.

Estimate the sum of the first 100,000 terms of the harmonic series,

$$\sum_{k=1}^{100,000}\frac{1}{k},$$

()
to the closest integer.
[Hint: Use left- and right-hand sums of the function $f(x) = 1/x$ on the interval from 1 to 100,000 with $\Delta x = 1$.]

ANSWER

WORKED SOLUTION

98.
Is the following argument true or false? Give reasons for your answer.

Consider the infinite series $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$. Since $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$ we can write this series as

$$\sum_{n=2}^{\infty} \frac{1}{n-1} - \sum_{n=2}^{\infty} \frac{1}{n}.$$

()
For the first series $a_n = 1/(n-1)$. Since $n - 1 < n$ we have $1/(n-1) > 1/n$ and so this series diverges by comparison with the divergent harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$. The second series is the divergent harmonic series. Since both series diverge, their difference also diverges.