Series Mirabilis

In 1693 Johann Bernoulli produced what he called the "Series Mirabilis" or wonderful (miraculous?) series. By applying Taylor series to the integral $\int_0^1 x^x \, \mathrm{d}x$ he discovered a fascinating result:

$$\int_0^1 x^x \, \mathrm{d}x = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots$$

The purpose of this project is to reproduce Bernoulli's discovery.

The first step in the process is to substitute the familiar exponential form of the integrand: $x^x = e^{x \ln x}$. Since we know the Taylor series for $y = e^x$ already, we can then substitute the appropriate values into the series.

In this project, you should generate the first four terms of the Taylor series for the integral above as well as an expression for its n^{th} term. Then evaluate the series for the values in the bounds of integration shown above. You will find it useful to reference the fact that (through L'hospital), $\lim_{x \to 0} x^a (\ln x)^b = 0, \quad (a \geq b)$.

Derive the rule for the n^{th} term in two ways:
(1) by following the pattern established by the first four terms of the series,
(2) by integrating the n^{th} term of the series you wrote for $e^{x \ln x}$.

It is worth noting that the terms in the Taylor series require, in some cases, extensive integration by parts resulting in small series for each term in the Taylor series - a series of series, in short. The Taylor series, therefore, has the form $\sum_{n=0}^{\infty} \sum_{k=0}^{n} f(k, n)$.