1. Use calculus to find the area of an ellipse with the formula \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

**Solution:**
\[
x = a\sqrt{1 - \frac{y^2}{b^2}} = \frac{a}{b}\sqrt{b^2 - y^2}
\]
The width of the rectangular strip is \( 2\frac{a}{b}\sqrt{b^2 - y^2} \).
The area of one strip of height \( \Delta y \) is therefore \( 2\frac{a}{b}\sqrt{b^2 - y^2} \Delta y \).

\[
\text{Area} = \int_{-b}^{b} 2\frac{a}{b}\sqrt{b^2 - y^2} \, dy = \frac{4a}{b} \int_{0}^{b} \sqrt{b^2 - y^2} \, dy
\]

2. Find the volume of the solid generated by revolving the area bounded by \( y = \frac{1}{3}x^2 \) between \( y = 0, y = 5 \), and the \( y \)-axis about the \( y \)-axis.

**Solution:**
\( x = \sqrt{3y} \) so 
\[
V = \int_{0}^{5} \pi (\sqrt{3y})^2 \, dy = \frac{3\pi}{2} y^2 \bigg|_{0}^{5} = \frac{75}{2} \pi
\]

3. Find the volume of the solid generated by revolving the area bounded by \( y = \sqrt{x} \) between \( x = 0, x = 4 \), and the \( x \)-axis about the \( y \)-axis.

**Solution:**
The outer radius is \( x = 4 \) while the inner radius is \( x = y^2 \).

Then the area of a washer slice is given by 
\[
\Delta A = \pi 4^2 - \pi (y^2)^2
\]
When \( x = 4, y = 2 \) so 
\[
V = \int_{0}^{2} \pi (16 - y^4) \, dy = \pi (16y - \frac{1}{5}y^5) \bigg|_{0}^{2} = \frac{128}{5} \pi
\]

4. Repeat #3 for the volume of the solid generated by revolving the area bounded by \( y = \sqrt{x} \) between \( x = 0, x = 4 \), and the \( x \)-axis about the axis \( x = 4 \).

**Solution:** In this case the radius of each disc is \( r = 4 - y^2 \).

Then 
\[\Delta A = \pi (4 - y^2)^2 \]
so 
\[
V = \int_{0}^{2} \pi (4 - y^2)^2 \, dy = \pi \int_{0}^{2} 16 - 8y^2 + y^4 \, dy
\]
\[
= \pi (16y - \frac{8}{3}y^3 + \frac{1}{5}y^5) \bigg|_{0}^{2} = \frac{256}{15} \pi
\]

5. Find the volume of the solid shown to the right.

**Solution:**
Since \( x = \frac{1}{y} \) it follows that the area of each disc is 
\[\Delta A = \pi \left(\frac{1}{y}\right)^2\].

From the bottom of the curve at \( y = \frac{1}{4} \) up to the top at \( y = 1 \) we have a volume of \( \int_{1/4}^{1} \pi \frac{1}{y^2} \, dy \). We also need to consider the volume of the cylinder beneath the caved shape. Its height is \( \frac{1}{4} \) and its radius is \( r = 4 \) so it has a volume of \( \pi 4^2 \cdot \frac{1}{4} = 4\pi \).

The total volume, then is
\[
V = \int_{1/4}^{1} \pi \frac{1}{y^2} \, dy + 4\pi = \pi - \pi \bigg|_{1/4}^{1} = 7\pi
\]
6. Derive the formula for the volume of a frustum where the base radii are $r_1$ and $r_2$ and the height is $h$.

**Solution:** From the first triangle we have

\[
\frac{r_2}{a} = \frac{r_1}{a - h} \quad \text{and} \quad r_2(a - h) = a \cdot r_1
\]

From the second triangle, we have

\[
\frac{r_2}{a} = \frac{x}{a - y} \quad \text{and} \quad x = \frac{r_2(a - y)}{a}
\]

Combining the two results we have: $x = \frac{hr_2 - y(r_2 - r_1)}{h}$

The volume of the frustum comes from

\[
V = \frac{\pi}{h^2} \int_0^h [hr_2 - y(r_2 - r_1)]^2 \, dy
\]

So $V = \frac{1}{3} \pi h (r_1^2 + r_1 r_2 + r_2^2)$

7. The region $R$ bounded by $f(x) = e^{-x^2}$ and $g(x) = 1 - \cos x$ is shown. Write the integral for the volume of the solid generated by revolving $R$ about the $x$-axis.

**Solution:** Let $b$ be the solution to $e^{-x^2} = 1 - \cos x$: $b \approx 0.9419$.

Then $V = \pi \int_0^b (e^{-x^2})^2 - (1 - \cos x)^2 \, dx$

8. An ice cream cone has radius 2.5 cm at the top. If a scoop of ice cream in the form of a sphere with radius 4 cm is placed on top of the cone, what percentage of the ice cream is outside the cone?

**Solution:** The integral looks much like the one for determining the volume of a sphere. (See the first sphere.) The only issue is the bounds of integration and these can be seen in the second figure.

\[
V = \pi \int_{-\sqrt{9.75}}^{\sqrt{9.75}} \left(16 - h^2 \right) dh = \pi \left(16h - \frac{1}{3} h^3\right)_{-\sqrt{9.75}}^{\sqrt{9.75}} \approx 259.11
\]

The percentage is given by dividing by the volume of the sphere $(4/3\pi(4)^3)$. We get about 97%.