1. Use calculus to find the area of an ellipse with the formula \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

**Solution:**

\[ x = a \sqrt{1 - \frac{y^2}{b^2}} = \frac{a}{b} \sqrt{b^2 - y^2} \]

so the width of the rectangular strip is \( \frac{2a}{b} \sqrt{b^2 - y^2} \).

The area of one strip of height \( \Delta y \) is therefore \( \frac{2a}{b} \sqrt{b^2 - y^2} \Delta y \).

Area = \( \int_{-b}^{b} \frac{2a}{b} \sqrt{b^2 - y^2} \, dy = \frac{4a}{b} \int_{0}^{b} \sqrt{b^2 - y^2} \, dy \)

2. Find the volume of the solid generated by revolving the area bounded by \( y = \frac{1}{3}x^2 \) between \( y = 0 \), \( y = 5 \), and the \( y \)-axis about the \( y \)-axis.

**Solution:**

\[ x = \sqrt{3y} \] so

\[ V = \int_{0}^{5} \pi (\sqrt{3y})^2 \, dy = \frac{3\pi}{2} y\sqrt{b^2 - y^2} \bigg|_{0}^{b} = \frac{75}{2} \pi \]

3. Find the volume of the solid generated by revolving the area bounded by \( y = \sqrt{x} \) between \( x = 0 \), \( x = 4 \), and the \( x \)-axis about the \( y \)-axis.

**Solution:**

The outer radius is \( x = 4 \) while the inner radius is \( x = y^2 \).

Then the area of a washer slice is given by \( \Delta A = \pi 4^2 - \pi (y^2)^2 \).

When \( x = 4, \ y = 2 \) so

\[ V = \int_{0}^{2} \pi (16 - y^4) \, dy = \pi (16y - \frac{1}{5} y^5) \bigg|_{0}^{2} = \frac{128}{5} \pi \]

4. Repeat #3 for the volume of the solid generated by revolving the area bounded by \( y = \sqrt{x} \) between \( x = 0 \), \( x = 4 \), and the \( x \)-axis about the axis \( x = 4 \).

**Solution:**

In this case the radius of each disc is \( r = 4 - y^2 \).

Then \( \Delta A = \pi 4 - \pi 4\sqrt{4 - y^2} \) so

\[ V = \int_{0}^{2} \pi (4 - y^2)^2 \, dy = \pi \int_{0}^{2} 16 - 8y^2 + y^4 \, dy = \pi (16y - \frac{8}{3} y^3 + \frac{1}{5} y^5) \bigg|_{0}^{2} = \frac{256}{15} \pi \]

5. Find the volume of the solid shown to the right.

**Solution:**

Since \( x = \frac{1}{y} \) it follows that the area of each disc is \( \Delta A = \pi \left( \frac{1}{y} \right)^2 \). From the bottom of the curve at \( y = \frac{1}{4} \) up to the top at \( y = 1 \) we have a volume of \( \int_{1/4}^{1} \pi \frac{1}{y^4} \, dy \). We also need to consider the volume of the cylinder beneath the caved shape. Its height is \( \frac{1}{4} \) and its radius is \( r = 4 \) so it has a volume of \( \pi 4^2 \cdot \frac{1}{4} = 4\pi \).

The total volume, then is

\[ V = \int_{1/4}^{1} \pi \frac{1}{y^2} \, dy + 4\pi = 4\pi - \pi \frac{1}{y} \bigg|_{1/4}^{1} = 7\pi. \]
6. Derive the formula for the volume of a frustum where the base radii are \( r_1 \) and \( r_2 \) and the height is \( h \).

**Solution:** From the first triangle we have

\[
\frac{r_2}{a} = \frac{r_1}{a - h}
\]

\[
r_2(a - h) = a \cdot r_1
\]

\[
a = \frac{hr_2}{r_2 - r_1}
\]

From the second triangle, we have

\[
\frac{r_2}{a} = \frac{x}{a - y}
\]

\[
x = \frac{r_2(a - y)}{a}
\]

Combining the two results we have: \( x = \frac{hr_2 - y(r_2 - r_1)}{h} \)

The volume of the frustum comes from

\[
V = \frac{\pi}{h^2} \int_0^h [hr_2 - y(r_2 - r_1)]^2 \, dy
\]

So \( V = \frac{1}{3} \pi h (r_1^2 + r_1 r_2 + r_2^2) \)

7. The region \( R \) bounded by \( f(x) = e^{-x^2} \) and \( g(x) = 1 - \cos x \) is shown. Write the integral for the volume of the solid generated by revolving \( R \) about the \( x \)-axis.

**Solution:** Let \( b \) be the solution to \( e^{-x^2} = 1 - \cos x \): \( b \approx 0.9419 \).

Then \( V = \pi \int_0^b (e^{-x^2})^2 - (1 - \cos x)^2 \, dx \)

8. An icecream cone has radius 2.5 cm at the top. If a scoop of icecream in the form of a sphere with radius 4 cm is placed on top of the cone, what percentage of the icecream is outside the cone?

**Solution:** The integral looks much like the one for determining the volume of a sphere. (See the first sphere.) The only issue is the bounds of integration and these can be seen in the second figure.

\[
V = \pi \int_{-\sqrt{16-h^2}}^4 16 - h^2 \, dx
\]