Trig Substitution Example

Find

$$\int \sqrt{1+x^2}\,\mathrm{d}x$$

Let

$$x = \tan\theta$$
$$\mathrm{d}x = \sec^2\theta\mathrm{d}\theta$$

Recall $\tan^2\theta + 1 = \sec^2\theta$

Then we have

$$\begin{aligned}\int \sqrt{1+x^2}\,\mathrm{d}x &= \int \sqrt{1+\tan^2\theta}\cdot\sec^2\theta\,\mathrm{d}\theta\\ &= \int \sqrt{\sec^2\theta}\cdot\sec^2\theta\,\mathrm{d}\theta\\ &= \int \sec^3\theta\,\mathrm{d}\theta\end{aligned}$$

We now have to find $\int \sec^3\theta\,\mathrm{d}\theta$.

Recall that $\mathrm{d}/\mathrm{d}x \tan x = \sec^2 x$ or $\int \sec^2 x = \tan x$
Then integrating by parts we have

$u = \sec\theta$ and $v' = \sec^2\theta$

$u' = \sec x\tan\theta$ and $v = \tan\theta$

$$\begin{aligned}\int \sec^3\theta\,\mathrm{d}\theta &= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,\mathrm{d}\theta\\ &= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\,\mathrm{d}\theta\\ &= \sec\theta\tan\theta - \int \sec^3\theta\,\mathrm{d}\theta + \int \sec\theta\,\mathrm{d}\theta\end{aligned}$$

Then

$$\begin{aligned}2\int \sec^3\theta\,\mathrm{d}\theta &= \sec\theta\tan\theta + \int \sec\theta\,\mathrm{d}\theta\\ \int \sec^3\theta\,\mathrm{d}\theta &= \frac{1}{2}\left(\sec\theta\tan\theta + \int \sec\theta\,\mathrm{d}\theta\right)\end{aligned}$$

Now we need to find $\int \sec\theta\,\mathrm{d}\theta$

$$\begin{aligned}\int \sec\theta\,\mathrm{d}\theta &= \int \sec\theta\cdot\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,\mathrm{d}\theta\\ &= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta+\tan\theta}\,\mathrm{d}\theta\end{aligned}$$

Let $u = \sec\theta + \tan\theta$
then $\mathrm{d}u = \sec\theta\tan\theta + \sec^2\theta\,\mathrm{d}\theta$

$$\begin{aligned}\int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta+\tan\theta}\,\mathrm{d}\theta &= \int \frac{\mathrm{d}u}{u}\\ &= \ln|u|\\ &= \ln|\sec\theta+\tan\theta|\end{aligned}$$

$$= \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)$$

From $x = \tan\theta$ and $\tan^2\theta + 1 = \sec^2\theta$ we have
$\sec\theta = \sqrt{1+x^2}$ We conclude

$$\boxed{\int \sqrt{1+x^2}\,\mathrm{d}x = \frac{1}{2}\left(x\sqrt{1+x^2} + \ln\left|x+\sqrt{1+x^2}\right|\right) + C}$$