The Directional Derivative

The derivative of a real valued function (scalar field) with respect to a vector.

What is the vector space analog to the usual derivative in one variable? $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$?

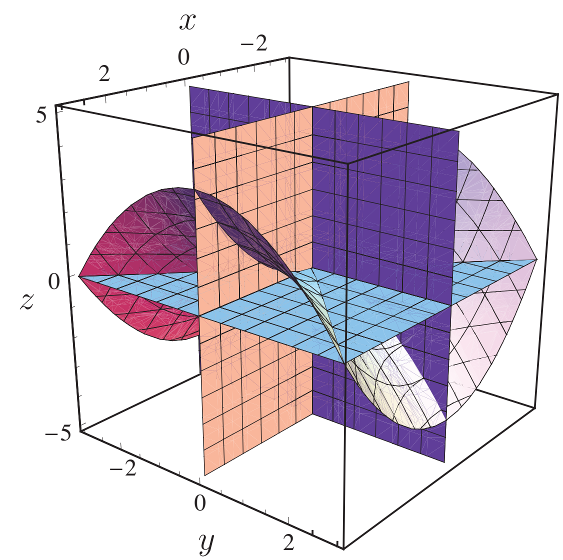

Figure 1: $f(x,y) = x^2 - y^2$

Suppose f is a real valued function (a mapping $f : \mathbb{R}^n \to \mathbb{R}$).
(e.g. $f(x,y) = x^2 - y^2$).

Unlike the case with plane figures, functions grow in a variety of ways at each point on a surface (or n–dimensional structure).

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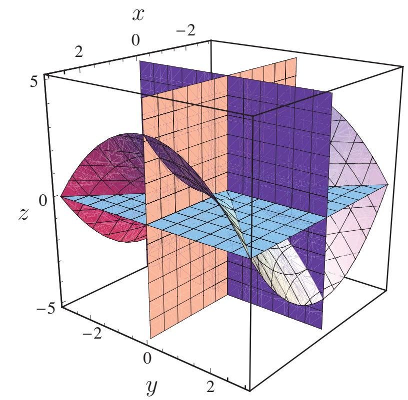

Figure 2: $f(x,y) = x^2 - y^2$

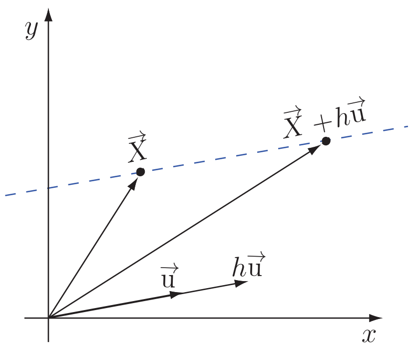

Figure 3: Change in position along line parallel to $\vec{u}$

If we give the direction by using a second vector, say $\vec{u}$, then for any real number h, the vector $\vec{X} + h\vec{u}$ represents a change in position from $\vec{X}$ along a line through $\vec{X}$ parallel to $\vec{u}$.

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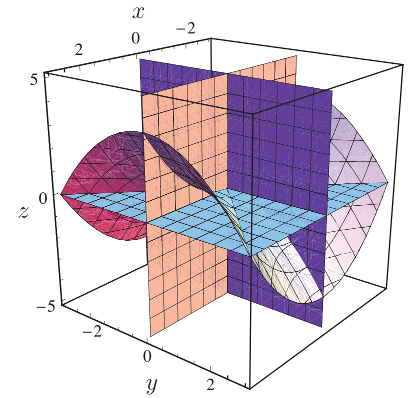

Figure 4: $f(x, y) = x^2 - y^2$

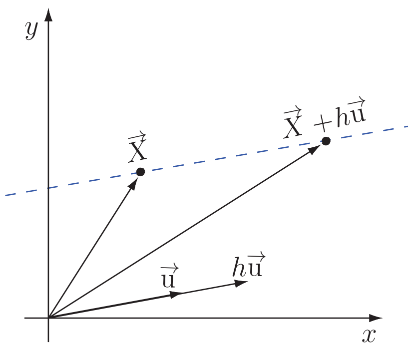

Figure 5: Change in position along line parallel to $\vec{u}$

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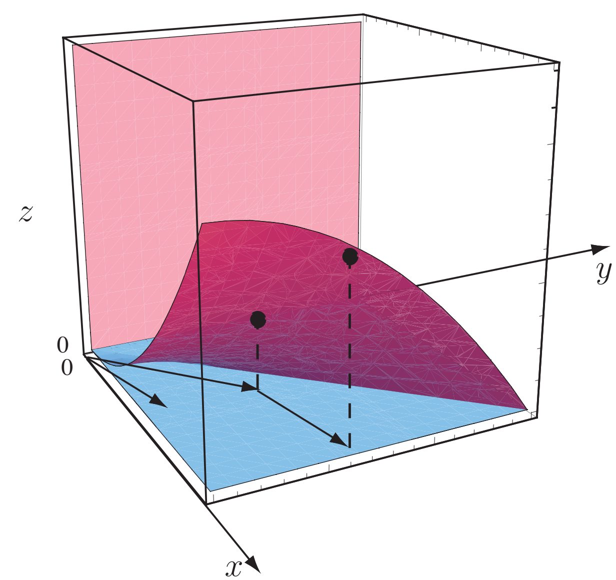
e.g. Imagine this diagram applied to $f(x, y) = x^2 - y^2$ and note the change in f as $\vec{X}$ changes along the line parallel to $\vec{u}$.



The Directional Derivative

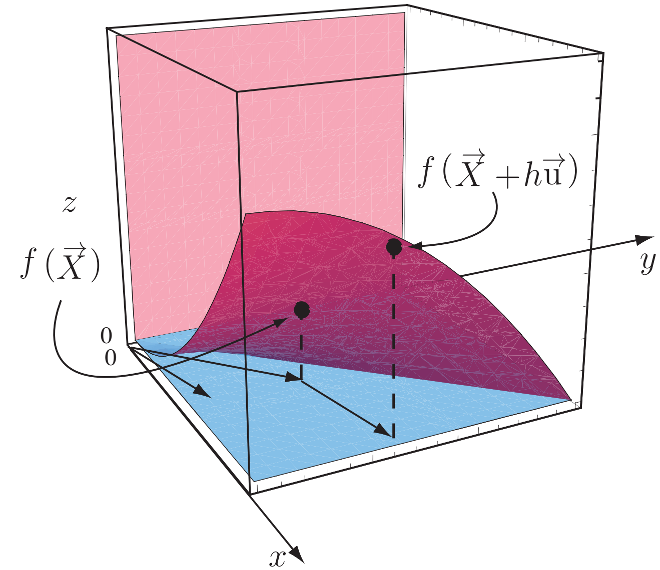


Figure 6: $f(x,y)=x^2-y^2$ in the first octant showing change from $\vec{X}$ to $\vec{X}+h\vec{u}$.

We can express this change as the difference quotient

$$\frac{f(\vec{X}+h\vec{u})-f(\vec{X})}{h}$$

which tells us the average rate of change in f over the segment from $\vec{X}$ to $\vec{X}+h\vec{u}$.

If we allow h to grow smaller and smaller the limiting value of the difference quotient (if it exists) is called the *Derivative of f in the direction of $\vec{u}$.*

If we add the condition that $\vec{u}$ is a unit vector then we get the special case:

Definition: The Directional Derivative of f at $\vec{X}$ with respect to unit vector $\vec{u}$ is defined by

$$f_{\vec{u}}(\vec{X})=f'(\vec{X};\vec{u})=\lim_{h\to 0}\frac{f(\vec{X}+h\vec{u})-f(\vec{X})}{h}$$

For a function with its domain in $\mathbb{R}^2$ we typically write

$$f_{\vec{u}}(x,y)=\lim_{h\to 0}\frac{f(x+hu_1,y+hu_2)-f(x,y)}{h}$$

Example:

$$f_{\vec{u}}(x,y) = \lim_{h\to 0} \frac{f(x+hu_1, y+hu_2) - f(x,y)}{h}$$

Consider the saddle curve $f(x,y) = x^2 - y^2$. Compute the rate of change in f at the point $(3,2)$ in the direction of the vector $\vec{Y} = 2\vec{i} + \vec{j}$.

First we need to construct a unit vector from $\vec{Y}$:

$$\vec{u} = \frac{\vec{Y}}{||\vec{Y}||} = \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}$$

Then

$$\begin{aligned}
f_{\vec{u}}(x,y) &= \lim_{h\to 0} \frac{f(x+hu_1, y+hu_2) - f(x,y)}{h} && (1)\\
&= \lim_{h\to 0} \frac{(x+hu_1)^2 - (y+hu_2)^2 - [x^2 - y^2]}{h} && (2)\\
&= \lim_{h\to 0} \frac{2xhu_1 + (hu_1)^2 - 2yhu_2 - (hu_2)^2}{h} && (3)\\
&= \lim_{h\to 0} \frac{\not{h}(2xu_1 + hu_1^2 - 2yu_2 - hu_2^2)}{\not{h}} && (4)\\
&= 2xu_1 - 2yu_2 && (5)
\end{aligned}$$

Evaluated at $(3,2)$ we have $f_{\vec{u}}(3,2) = 6\left(\frac{2}{\sqrt{5}}\right) + -4\left(\frac{1}{\sqrt{5}}\right) = \frac{8}{\sqrt{5}}$

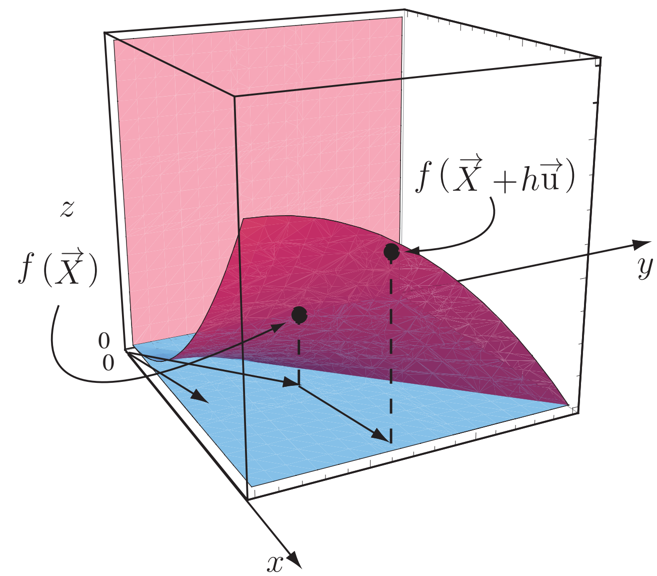


Figure 7: $f(x,y) = x^2 - y^2$ in the first octant.

$$f_{\vec{u}}(x,y) = 6\left(\frac{2}{\sqrt{5}}\right) + -4\left(\frac{1}{\sqrt{5}}\right) = \frac{8}{\sqrt{5}}$$

Two things worth noting from the previous example are,

(1) The result of the derivative $f_{\vec{u}}(x,y) = 2xu_1 - 2yu_2$ is equivalent to $\dfrac{\partial f(x,y)}{\partial x}u_1 + \dfrac{\partial f(x,y)}{\partial y}u_2$ which is suggestive of a significant consequence of the definition.

(2) The significance of the unit vector becomes clearer if we consider units for this example. Suppose $f(x,y)$ gives temperature (in degrees) at each location (x,y) and we're interested in how quickly the temperature changes in a particular direction. If we are measuring distance in meters then the units of our rate of change are degrees per meter.
If we left $\vec{Y}$ in the previous example then our rate would be 8 but measured in units $||\vec{Y}|| = \sqrt{5}$ m long.

A convenient consequence of the Mean Value Theorem (or equivalently of the linear property of the derivative we've defined) is the relationship alluded to in the previous example:

$$f_{\vec{\mathrm{u}}}(\vec{X}) = \lim_{h \to 0} \frac{f(\vec{X} + h\vec{\mathrm{u}}) - f(\vec{X})}{h} = \sum_{k=1}^{n} \frac{\partial f(\vec{X})}{\partial x_k} u_k$$

Where $\vec{X} = (x_1, x_2, \ldots, x_n)$ and $\vec{\mathrm{u}} = (u_1, u_2, \ldots, u_n)$.

For a function with its domain in $\mathbb{R}^2$ we typically write

$$\begin{aligned} f_{\vec{\mathrm{u}}}(x, y) = \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h} &= \frac{\partial f(x, y)}{\partial x} u_1 + \frac{\partial f(x, y)}{\partial y} u_2 \\ &= f_x(x, y) u_1 + f_y(x, y) u_2 \end{aligned}$$

and for $f : \mathbb{R}^3 \to \mathbb{R}$

$$\begin{aligned} f_{\vec{\mathrm{u}}}(x, y, z) = \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2, z + hu_3) - f(x, y, z)}{h} &= \frac{\partial f(x, y, z)}{\partial x} u_1 + \frac{\partial f(x, y, z)}{\partial y} u_2 + \frac{\partial f(x, y,}{\partial z} \\ &= f_x(x, y, z) u_1 + f_y(x, y, z) u_2 + f_z(x, y, z) \end{aligned}$$

We'll return to the proof of this when we discuss the chain rule.

Gradient

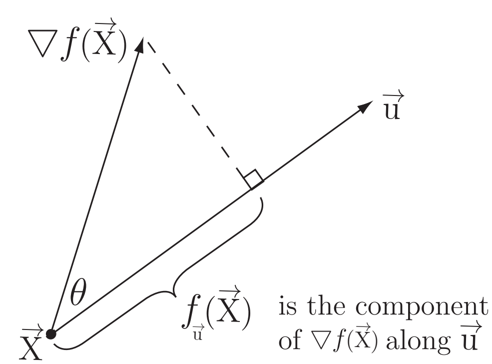
$$f_{\vec{u}}(x,y,z) = \frac{\partial f(x,y,z)}{\partial x}u_1 + \frac{\partial f(x,y,z)}{\partial y}u_2 + \frac{\partial f(x,y,z)}{\partial z}u_3$$

$$= \left(\frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z}\right) \cdot (u_1, u_2, u_3)$$

This relationship reflects the result of a dot product between the vector $\vec{u} = (u_1, u_2, u_3)$ and the vector of the partial derivatives of f. For this and many other reasons of convenience and insight we introduce the *Gradient*:

Definition: The Gradient of a real valued function, f, at a point $\vec{X}$ is denoted $\nabla f(\vec{X})$ and represents a vector field whose value at $\vec{X}$ is given by:

$$\nabla f(\vec{X}) = \left(\frac{\partial f(\vec{X})}{\partial x_1}, \frac{\partial f(\vec{X})}{\partial x_2}, \ldots, \frac{\partial f(\vec{X})}{\partial x_n}\right)$$



In the case of $f : \mathbb{R}^3 \to \mathbb{R}$ we write

$$\nabla f(x,y,z) = \left(\frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z}\right) = \frac{\partial f(x,y,z)}{\partial x}\vec{i} + \frac{\partial f(x,y,z)}{\partial y}\vec{j} + \frac{\partial f(x,y,z)}{\partial z}\vec{k}$$

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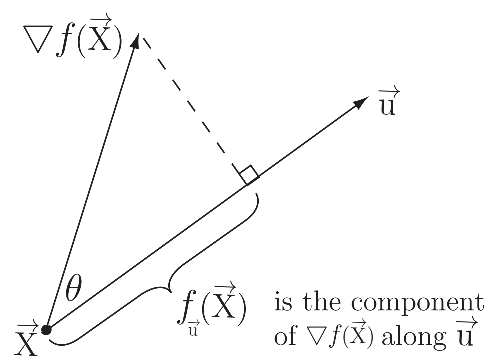

$$\nabla f(\vec{X}) = \left(\frac{\partial f(\vec{X})}{\partial x_1}, \frac{\partial f(\vec{X})}{\partial x_2}, \ldots, \frac{\partial f(\vec{X})}{\partial x_n}\right)$$

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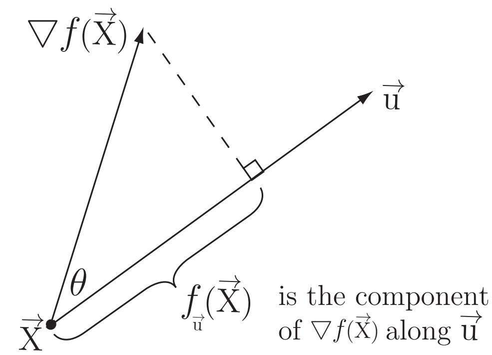
It follows that $f_{\vec{u}}(\vec{X}) = \nabla f(\vec{X}) \cdot \vec{u}$

Gradient

From the geometric interpretation of the dot product we have

$$\begin{aligned} f_{\vec{\mathrm{u}}}(\vec{X}) &= \nabla f(\vec{X}) \cdot \vec{\mathrm{u}} \\ &= ||\nabla f(\vec{X})||||\vec{\mathrm{u}}|| \cos\theta \\ &= ||\nabla f(\vec{X})|| \cos\theta \end{aligned}$$



Meaning that the directional derivative is the component of the gradient vector in the direction of the unit vector $\vec{\mathrm{u}}$.

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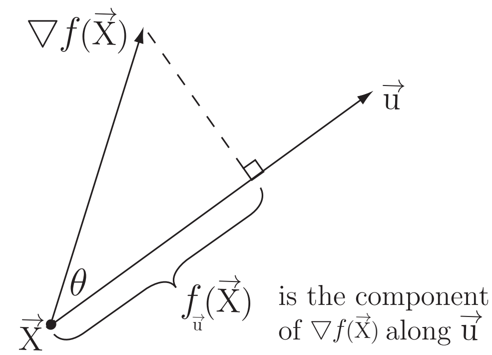

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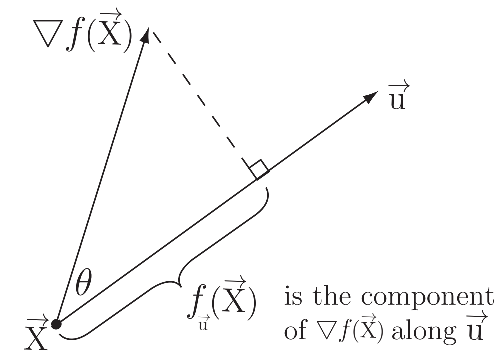

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(1) The directional derivative $f_{\vec{u}}(\vec{X})$ is at its maximum when $\theta = 0$ (so when $\vec{u}$ is in the direction of ∇f). That is, at a given point $\vec{X}$, the function f undergoes its greatest rate of change in the direction of the gradient vector.

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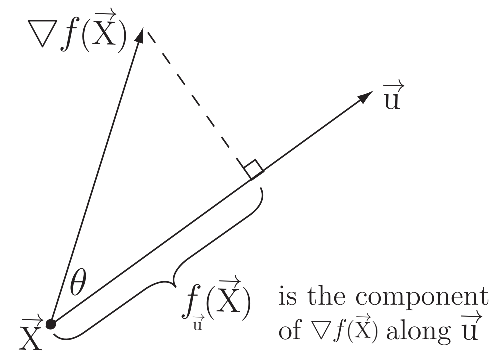

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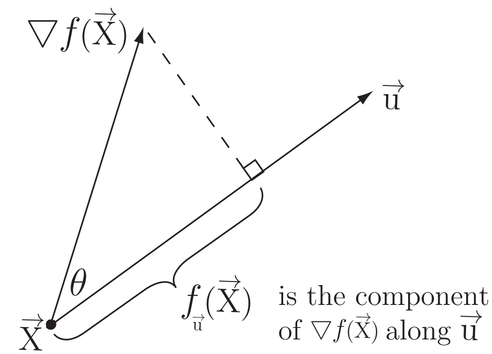

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(3) When $\vec{\mathrm{u}}$ is perpendicular to ∇f the directional derivative $f_{\vec{\mathrm{u}}}(\vec{X}) = 0$.

Example:

In what direction from $(3,2)$ does $f(x,y) = x^2 - y^2$ increase the fastest?

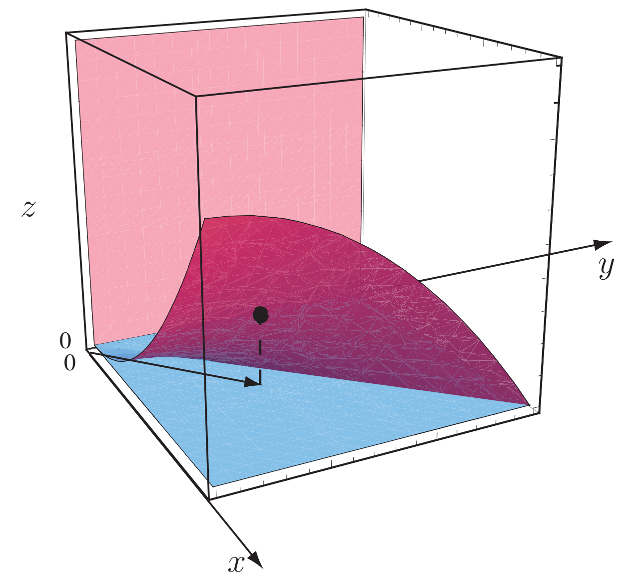

Figure 8: $f(x,y) = x^2 - y^2$ in the first octant.

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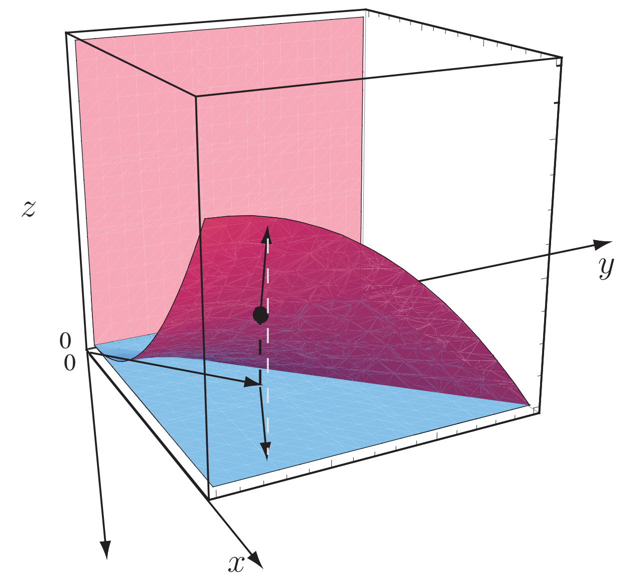

Figure 9: $f(x,y) = x^2 - y^2$ in the first octant.

Solution: The gradient $\nabla f(x,y) = 2x\vec{i} - 2y\vec{j}$ at $(3,2)$ is $\nabla f = 6\vec{i} - 4\vec{j}$ which points in the direction of the greatest rate of climb.

Note the translation of the vector to the surface of the saddle.

Example:

1. Discuss the sign (+), (–) or 0 of the directional derivative for $f(x,y) = x^2 - y^2$ at the various points indicated.

(1) A in the direction $\vec{i}$

(2) A in the direction $\vec{j}$

(3) A in the direction $\vec{i} + \vec{j}$

(4) B in the direction $\vec{i}$

(4) B in the direction $\vec{-i}$

(5) B in the direction $\vec{j}$

(6) C in the direction $\vec{j}$

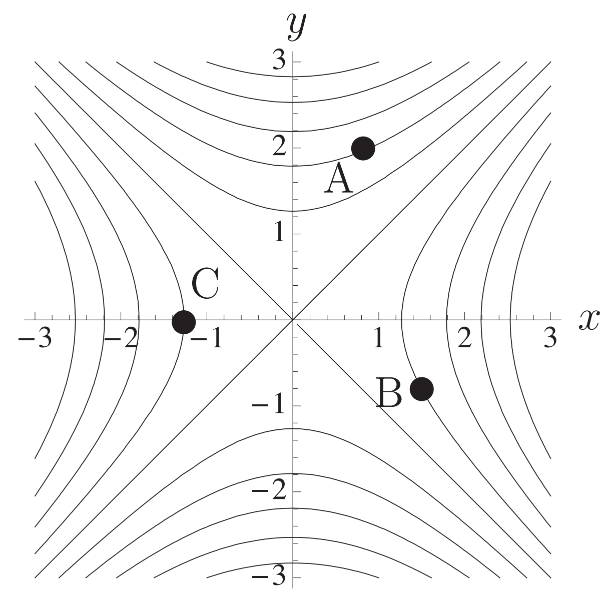

Figure 10: Level curves of $f(x, y) = x^2 - y^2$.

2. Estimate the gradients at each point.

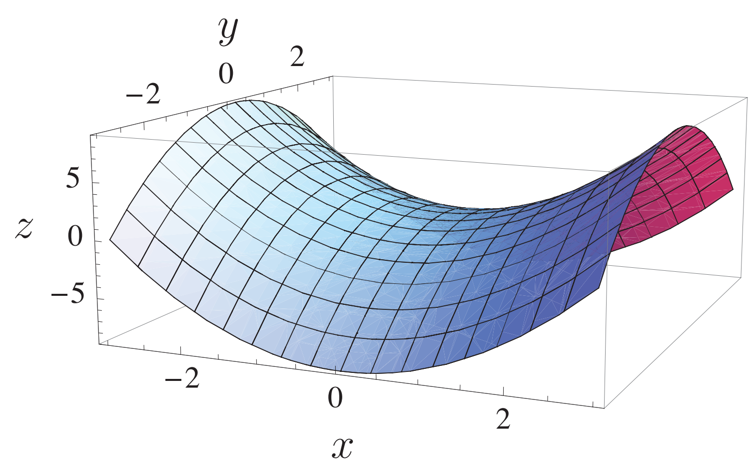

Figure 11: $f(x, y) = x^2 - y^2$.

Chain Rule (Special case in $\mathbb{R}^3$):

Let $\mathbf{v} : \mathbb{R} \to \mathbb{R}^3$ be a differentiable path and $f : \mathbb{R}^3 \to \mathbb{R}$ a real valued function. Suppose $h(t) = f(\mathbf{v}(t)) = f(x(t), y(t), z(t))$ where $\mathbf{v}(t) = (x(t), y(t), z(t))$. Then

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{\partial f(x,y,z)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f(x,y,z)}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f(x,y,z)}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= \nabla f(\mathbf{v}(t)) \cdot \mathbf{v}'(t)\end{aligned}$$

Where $\mathbf{v}'(t) = (x'(t), y'(t), z'(t))$

Proof:

$$h'(t_0) = \lim_{t \to t_0} \frac{h(t) - h(t_0)}{t - t_0}$$

$$\begin{aligned}\frac{h(t) - h(t_0)}{t - t_0} &= \frac{f(x(t), y(t), z(t)) - f(x(t_0), y(t_0), z(t_0))}{t - t_0} \\ &= \frac{f(x(t), y(t), z(t)) - f(x(t_0), y(t), z(t))}{t - t_0} \\ &\quad + \frac{f(x(t_0), y(t), z(t)) - f(x(t_0), y(t_0), z(t))}{t - t_0} \\ &\quad + \frac{f(x(t_0), y(t_0), z(t)) - f(x(t_0), y(t_0), z(t_0))}{t - t_0}\end{aligned}$$

$$
\begin{aligned}
\frac{h(t)-h(t_0)}{t-t_0} &= \frac{f(x(t),y(t),z(t))-f(x(t_0),y(t_0),z(t_0))}{t-t_0} \\
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&\quad + \frac{f(x(t_0),y(t_0),z(t))-f(x(t_0),y(t_0),z(t_0))}{t-t_0}
\end{aligned}
$$

From the Mean Value Theorem it follows that there is some $c \in (x, x_0)$ such that

$$
\frac{f(x,y,z)-f(x_0,y,z)}{x-x_0} = \frac{\partial f(c,y,z)}{\partial x} \longrightarrow f(x,y,z)-f(x_0,y,z) = \frac{\partial f(c,y,z)}{\partial x}(x-x_0)
$$

$$
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Then similarly for $c \in (x(t), x(t_0))$, $d \in (y(t), y(t_0))$ and $e \in (z(t), z(t_0))$

It follows that

$$
\frac{h(t)-h(t_0)}{t-t_0} = \left(\frac{\partial f(c,y(t),z(t))}{\partial x}\right)\frac{x(t)-x(t_0)}{t-t_0} + \left(\frac{\partial f(x(t_0),d,z(t))}{\partial y}\right)\frac{y(t)-y(t_0)}{t-t_0} + \left(\frac{\partial f(x(t_0),y(t_0),e)}{\partial z}\right)\frac{z(t)-z(t_0)}{t-t_0}
$$

$$\begin{aligned}\frac{h(t)-h(t_0)}{t-t_0} &= \frac{f(x(t),y(t),z(t))-f(x(t_0),y(t_0),z(t_0))}{t-t_0}\\ &= \frac{f(x(t),y(t),z(t))-f(x(t_0),y(t),z(t))}{t-t_0}\\ &\quad+\frac{f(x(t_0),y(t),z(t))-f(x(t_0),y(t_0),z(t))}{t-t_0}\\ &\quad+\frac{f(x(t_0),y(t_0),z(t))-f(x(t_0),y(t_0),z(t_0))}{t-t_0}\end{aligned}$$

From the Mean Value Theorem it follows that there is some $c \in (x, x_0)$ such that

$$\frac{f(x,y,z)-f(x_0,y,z)}{x-x_0} = \frac{\partial f(c,y,z)}{\partial x} \longrightarrow f(x,y,z)-f(x_0,y,z) = \frac{\partial f(c,y,z)}{\partial x}(x-x_0)$$

Then similarly for $c \in (x(t), x(t_0))$, $d \in (y(t), y(t_0))$ and $e \in (z(t), z(t_0))$

It follows that

$$\frac{h(t)-h(t_0)}{t-t_0} = \left(\frac{\partial f(c,y(t),z(t))}{\partial x}\right)\frac{x(t)-x(t_0)}{t-t_0}+\left(\frac{\partial f(x(t_0),d,z(t))}{\partial y}\right)\frac{y(t)-y(t_0)}{t-t_0}+\left(\frac{\partial f(x(t_0),y(t_0),e)}{\partial z}\right)\frac{z(t)-z(t_0)}{t-t_0}$$

As $t \to t_0$, $$= \left(\frac{\partial f(c,y(t),z(t))}{\partial x}\right)\frac{\mathrm{d}x}{\mathrm{d}t}+\left(\frac{\partial f(x(t_0),d,z(t))}{\partial y}\right)\frac{\mathrm{d}y}{\mathrm{d}t}+\left(\frac{\partial f(x(t_0),y(t_0),e)}{\partial z}\right)\frac{\mathrm{d}z}{\mathrm{d}t}$$

$$
\begin{aligned}
\frac{h(t)-h(t_0)}{t-t_0} &= \frac{f(x(t),y(t),z(t))-f(x(t_0),y(t_0),z(t_0))}{t-t_0} \\
&= \frac{f(x(t),y(t),z(t))-f(x(t_0),y(t),z(t))}{t-t_0} \\
&\quad + \frac{f(x(t_0),y(t),z(t))-f(x(t_0),y(t_0),z(t))}{t-t_0} \\
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\end{aligned}
$$

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$$

And as $t \to t_0$, we have $c \to x(t_o), d \to y(t_0), e \to z(t_0)$ we have

$$
\boxed{\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial f(x,y,z)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f(x,y,z)}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f(x,y,z)}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}}
$$

Directional Derivative (again)

In the case where $f : \mathbb{R}^3 \to \mathbb{R}$ we let $\vec{X} = (x, y, z)$ and claim that

$$\begin{aligned} f_{\vec{\mathrm{u}}}(x, y, z) &= \nabla f(x, y, z) \cdot \vec{\mathrm{u}} \\ &= \frac{\partial f(x, y, z)}{\partial x} u_1 + \frac{\partial f(x, y, z)}{\partial y} u_2 + \frac{\partial f(x, y, z)}{\partial z} u_3 \end{aligned}$$

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Proof:

Recall The Directional Derivative of f at $\vec{X}$ with respect to unit vector $\vec{\mathrm{u}}$ is defined by

$$f_{\vec{\mathrm{u}}}(\vec{X}) = f'(\vec{X}; \vec{\mathrm{u}}) = \lim_{h \to 0} \frac{f(\vec{X} + h\vec{\mathrm{u}}) - f(\vec{X})}{h}$$

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Let $\mathbf{v}(t) = \vec{X} + t\vec{\mathrm{u}}$ then $f(\vec{X} + t\vec{\mathrm{u}}) = f(\mathbf{v}(t))$

From the chain rule $\dfrac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{v}(t)) = \nabla f(\mathbf{v}(t)) \cdot \mathbf{v}'(t)$.

Since $\mathbf{v}(0) = \vec{X}$ and $\mathbf{v}'(0) = \vec{\mathrm{u}}$ we have $f_{\vec{\mathrm{u}}}(x,y,z) = \dfrac{\partial f(x,y,z)}{\partial x} u_1 + \dfrac{\partial f(x,y,z)}{\partial y} u_2 + \dfrac{\partial f(x,y,z)}{\partial z} u_3$ $\square$

$$f_{\vec{u}}(x, y, z) = \frac{\partial f(x, y, z)}{\partial x}u_1 + \frac{\partial f(x, y, z)}{\partial y}u_2 + \frac{\partial f(x, y, z)}{\partial z}u_3$$

Example: Let $f(x, y, z) = x^2e^{-yz}$. Find the rate of change in f in the direction of $\vec{u} = \frac{1}{\sqrt{3}}\vec{i} - \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}$ at the point (1, 0, 0).

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Solution: $\nabla f(x, y, z) = (2xe^{-yz}, -x^2ze^{-yz}, -x^2ye^{-yz}) \longrightarrow (2, 0, 0)$

So $f_{\vec{u}}(x, y, z) = \nabla f \cdot \vec{u} = (2xe^{-yz}, -x^2ze^{-yz}, -x^2ye^{-yz}) \cdot \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

And $f_{\vec{u}}(1, 0, 0) = (2, 0, 0) \cdot \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}$ $\square$

The Gradient is Perpendicular to Level Curves

First let's agree that there is reasonable cause to support this assertion. We define perpendicular to mean that the gradient is perpendicular to the line tangent to a curve at a specified point.

Recall from geometry that the shortest distance from a point P to a line, ℓ, lies on the line through P perpendicular to ℓ. Then the shortest distance between two level curves (and therefore the greatest rate of increase) should lie along the line that is in some sense perpendicular to one of the curves.

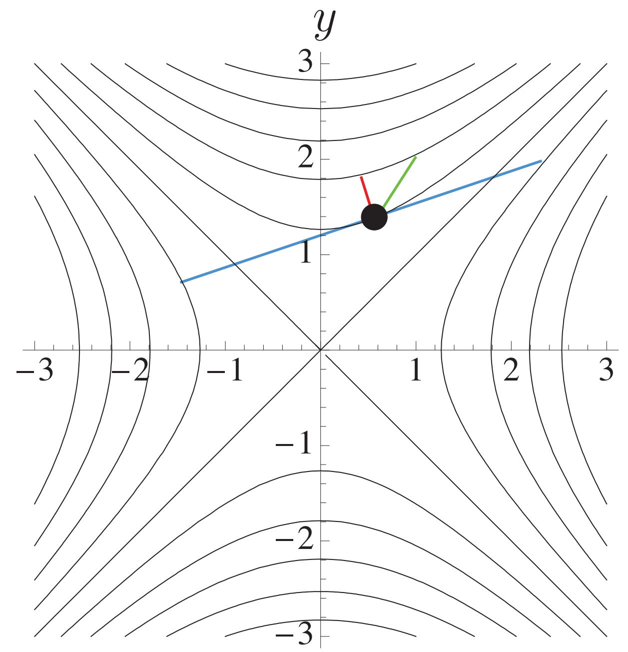

Figure 12: Level curves of $f(x, y) = x^2 - y^2$

The Gradient is Perpendicular to Level Curves

From the definition each level curve of a real valued function $f(x, y)$ has the form $f(x, y) = k$.
Let $\mathbf{c}(t)$ be a differentiable curve in the plane with an arbitrary level curve, L.
Let $\mathbf{v}$ be the vector tangent to $\mathbf{c}$ at $t = 0$ (so at the point $\mathbf{c}(0) = (x_0, y_0)$). Therefore $\mathbf{c}'(0) = \mathbf{v}$.

Then as we approach a level curve of f along $\mathbf{c}(t)$ we have $f(\mathbf{c}(t)) = k$ and therefore $\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{c}(t)) = 0$.

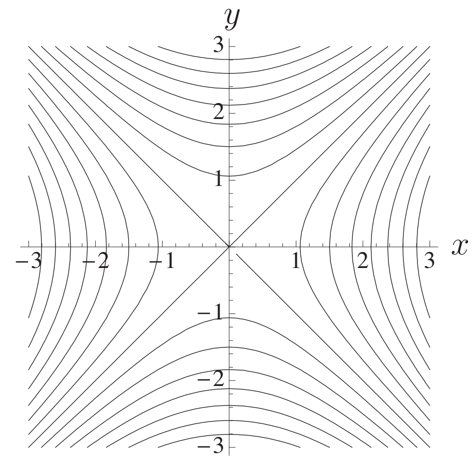

Figure 13: Level curves of $f(x, y) = x^2 - y^2$

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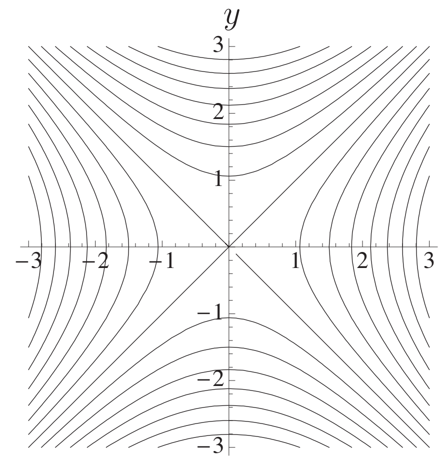

Figure 14: Level curves of $f(x, y) = x^2 - y^2$

Applying the chain rule we also have
$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$.
Evaluating at $t = 0$ this gives us $\nabla f(\mathbf{c}(0)) \cdot \mathbf{c}'(0) = \nabla f(\mathbf{c}(0)) \cdot \mathbf{v}$.
Equating the two halves gives us: $\nabla f(\mathbf{c}(0)) \cdot \mathbf{v} = 0$
So we conclude that the gradient ∇f at a point is perpendicular to the level curve of f at that point.

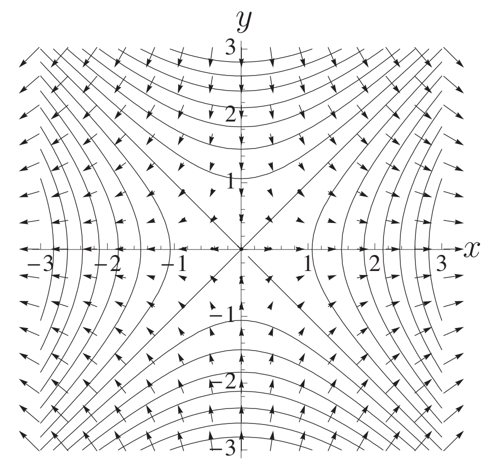

Figure 15: Gradient field of $f(x, y) = x^2 - y^2$

Note that since $\nabla f(\mathbf{c}(0)) \cdot \mathbf{v} = \nabla f(x_0, y_0) \cdot \mathbf{v} = 0$ we can construct the set of all points in the plane of the level curve that lie on the line tangent to the curve at (x_0, y_0):

$$\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0$$

Gradient Field Example

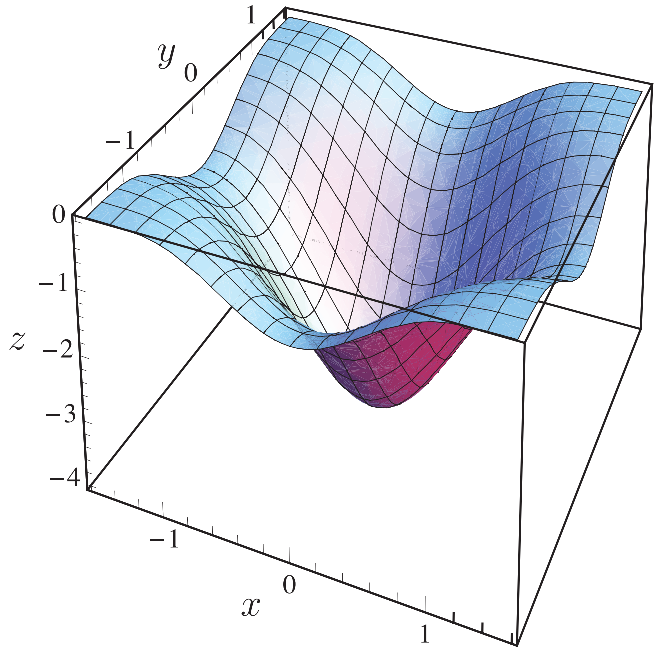

Figure 16: $f(x, y) = -(\cos^2 x + \cos^2 y)^2$

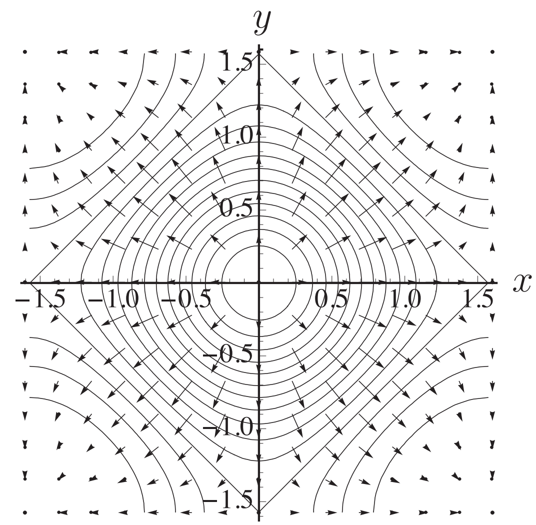

Figure 17: Contours and Gradient field $f(x, y) = -(\cos^2 x + \cos^2 y)^2$

Example:

1. Do the level curves of $f(x, y) = x + y$ cross the level curves of $g(x, y) = x - y$ at right angles?

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Solution:

If the level curve of f is perpendicular to a level curve of g then what must be true about their gradients at the point of intersection?

The Gradient is Normal to Level Surfaces

The argument in three variables is identical to that in two and we derive the conclusion that for a level surface, S, composed of points (x, y, z) where $f(x, y, z) = k$, $(k \in \mathbb{R})$, the tangent plane of S at a point (x_0, y_0, z_0) of S is given by

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Equivalently,

$$\frac{\partial f(x_0, y_0, z_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0, z_0)}{\partial y}(y - y_0) + \frac{\partial f(x_0, y_0, z_0)}{\partial z}(z - z_0) = 0$$

OR,

$$\frac{\partial f(x_0, y_0, z_0)}{\partial x}x + \frac{\partial f(x_0, y_0, z_0)}{\partial y}y + \frac{\partial f(x_0, y_0, z_0)}{\partial z}z + d = 0$$

Where $d = -\left(\frac{\partial f(x_0, y_0, z_0)}{\partial x}x_0 + \frac{\partial f(x_0, y_0, z_0)}{\partial y}y_0 + \frac{\partial f(x_0, y_0, z_0)}{\partial z}z_0\right)$

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Example:

Find the equation of the plane tangent to the surface defined by $3xy + z^2 = 4$ at $(1, 1, 1)$.

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Example:
Find the equation of the plane tangent to the surface defined by $3xy + z^2 = 4$ at $(1, 1, 1)$.

Solution:

$\nabla f(x, y, z) = (3y, 3x, 2z) \longrightarrow \nabla f(1, 1, 1) = (3, 3, 2)$

Then $(3, 3, 2) \cdot (x - 1, y - 1, z - 1) = 3x + 3y + 2z - 8$

So we have $3x + 3y + 2z = 8 \qquad \square$

BEWARE

For a function $f : \mathbb{R}^2 \to \mathbb{R}$, typically seen as $z = f(x, y)$

$\nabla f(x_0, y_0)$ gives the vector in the direction

- perpendicular to the level curve containing (x_0, y_0)
- of the greatest rate of change along the surface of $z = f(x, y)$ at (x_0, y_0).

$\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0$ gives the equation of the *line* tangent to the level curve $f(x, y) = k$ at (x_0, y_0).

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NOT the equation of the plane tangent to the surface $(x_0, y_0, f(x_0, y_0))$.

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Example:

Find the equation of the plane tangent to the curve $z = x^2 - y^2$ at $(3, 2)$.

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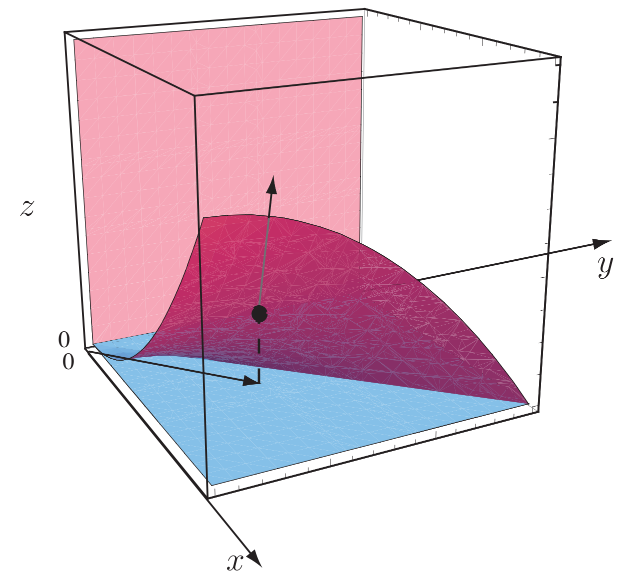

Figure 18: Normal vector to $f(x, y) = x^2 - y^2$ at (3,2,5)

Solution:

We know that the equation of the plane perpendicular to S at (x_0, y_0, z_0) is given by $\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$.

So we begin by writing $f(x, y, z) = z - x^2 + y^2$

(the curve $z = x^2 - y^2$ is the case where $f(x, y, z) = 0$)

$\nabla f(x, y, z) = (-2x, 2y, 1) \longrightarrow (-6, 4, 1)$

So the equation of the tangent plane at $(3, 2, 5)$ is $(-6, 4, 1) \cdot (x - x_0, y - y_0, z - z_0) = 0$

Or $-6x + 4y + z = -5$