1.3 SOLUTIONS

Notes: The key exercises are 11–14, 17–22, 25, and 26. A discussion of Exercise 25 will help students understand the notation $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3]$, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, and $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

1. $\mathbf{u}+\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1+(-3)\\2+(-1)\end{bmatrix}=\begin{bmatrix}-4\\1\end{bmatrix}$.

Using the definitions carefully,

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}+(-2)\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}(-2)(-3)\\(-2)(-1)\end{bmatrix}=\begin{bmatrix}-1+6\\2+2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$$, or, more quickly,

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}-2\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1+6\\2+2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$$. The intermediate step is often not written.

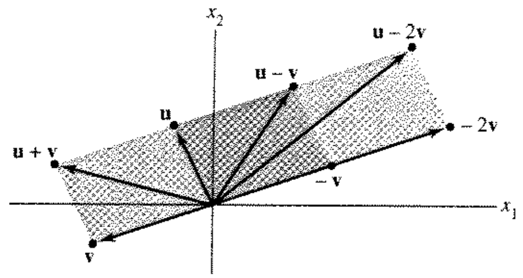
2. $\mathbf{u}+\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3+2\\2+(-1)\end{bmatrix}=\begin{bmatrix}5\\1\end{bmatrix}$.

Using the definitions carefully,

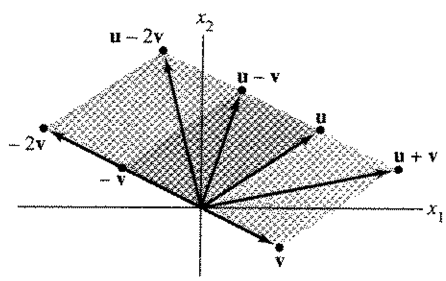
$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}+(-2)\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}(-2)(2)\\(-2)(-1)\end{bmatrix}=\begin{bmatrix}3+(-4)\\2+2\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$, or, more quickly,

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}-2\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3-4\\2+2\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$. The intermediate step is often not written.

3.



4.



5. $$x_1\begin{bmatrix}6\\-1\\5\end{bmatrix}+x_2\begin{bmatrix}-3\\4\\0\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix},\quad \begin{bmatrix}6x_1\\-x_1\\5x_1\end{bmatrix}+\begin{bmatrix}-3x_2\\4x_2\\0\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix},\quad \begin{bmatrix}6x_1-3x_2\\-x_1+4x_2\\5x_1\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix}$$

$$\begin{aligned}6x_1 - 3x_2 &= 1\\ -x_1 + 4x_2 &= -7\\ 5x_1 &= -5\end{aligned}$$

Usually the intermediate steps are not displayed.

6. $$x_1\begin{bmatrix}-2\\3\end{bmatrix}+x_2\begin{bmatrix}8\\5\end{bmatrix}+x_3\begin{bmatrix}1\\-6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad \begin{bmatrix}-2x_1\\3x_1\end{bmatrix}+\begin{bmatrix}8x_2\\5x_2\end{bmatrix}+\begin{bmatrix}x_3\\-6x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad \begin{bmatrix}-2x_1+8x_2+x_3\\3x_1+5x_2-6x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{aligned}-2x_2 + 8x_2 + x_3 &= 0\\ 3x_1 + 5x_2 - 6x_3 &= 0\end{aligned}$$

Usually the intermediate steps are not displayed.

7. See the figure below. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbf{R}^2$ can be written as a linear combination of **u** and **v**.

To write a vector **a** as a linear combination of **u** and **v**, imagine walking from the origin to **a** along the grid "streets" and keep track of how many "blocks" you travel in the **u**-direction and how many in the **v**-direction.

a. To reach **a** from the origin, you might travel 1 unit in the **u**-direction and –2 units in the **v**-direction (that is, 2 units in the negative **v**-direction). Hence **a** = **u** – 2**v**.

b. To reach **b** from the origin, travel 2 units in the **u**-direction and –2 units in the **v**-direction. So $\mathbf{b} = 2\mathbf{u} - 2\mathbf{v}$. Or, use the fact that **b** is 1 unit in the **u**-direction from **a**, so that

$$\mathbf{b} = \mathbf{a} + \mathbf{u} = (\mathbf{u} - 2\mathbf{v}) + \mathbf{u} = 2\mathbf{u} - 2\mathbf{v}$$

c. The vector **c** is –1.5 units from **b** in the **v**-direction, so

$$\mathbf{c} = \mathbf{b} - 1.5\mathbf{v} = (2\mathbf{u} - 2\mathbf{v}) - 1.5\mathbf{v} = 2\mathbf{u} - 3.5\mathbf{v}$$

d. The "map" suggests that you can reach **d** if you travel 3 units in the **u**-direction and –4 units in the **v**-direction. If you prefer to stay on the paths displayed on the map, you might travel from the origin to –3**v**, then move 3 units in the **u**-direction, and finally move –1 unit in the **v**-direction. So

$$\mathbf{d} = -3\mathbf{v} + 3\mathbf{u} - \mathbf{v} = 3\mathbf{u} - 4\mathbf{v}$$

Another solution is

$$\mathbf{d} = \mathbf{b} - 2\mathbf{v} + \mathbf{u} = (2\mathbf{u} - 2\mathbf{v}) - 2\mathbf{v} + \mathbf{u} = 3\mathbf{u} - 4\mathbf{v}$$

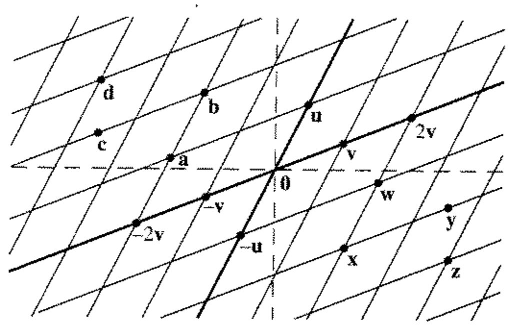


Figure for Exercises 7 and 8

8. See the figure above. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbf{R}^2$ can be written as a linear combination of **u** and **v**.

w. To reach **w** from the origin, travel –1 units in the **u**-direction (that is, 1 unit in the negative **u**-direction) and travel 2 units in the **v**-direction. Thus, $\mathbf{w} = (-1)\mathbf{u} + 2\mathbf{v}$, or $\mathbf{w} = 2\mathbf{v} - \mathbf{u}$.

x. To reach **x** from the origin, travel 2 units in the **v**-direction and –2 units in the **u**-direction. Thus, $\mathbf{x} = -2\mathbf{u} + 2\mathbf{v}$. Or, use the fact that **x** is –1 units in the **u**-direction from **w**, so that

$$\mathbf{x} = \mathbf{w} - \mathbf{u} = (-\mathbf{u} + 2\mathbf{v}) - \mathbf{u} = -2\mathbf{u} + 2\mathbf{v}$$

y. The vector **y** is 1.5 units from **x** in the **v**-direction, so

$$\mathbf{y} = \mathbf{x} + 1.5\mathbf{v} = (-2\mathbf{u} + 2\mathbf{v}) + 1.5\mathbf{v} = -2\mathbf{u} + 3.5\mathbf{v}$$

z. The map suggests that you can reach **z** if you travel 4 units in the **v**-direction and –3 units in the **u**-direction. So $\mathbf{z} = 4\mathbf{v} - 3\mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$. If you prefer to stay on the paths displayed on the "map," you might travel from the origin to –2**u**, then 4 units in the **v**-direction, and finally move –1 unit in the **u**-direction. So

$$\mathbf{z} = -2\mathbf{u} + 4\mathbf{v} - \mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$$

9. $$\begin{aligned} x_2 + 5x_3 &= 0 \\ 4x_1 + 6x_2 - x_3 &= 0, \\ -x_1 + 3x_2 - 8x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} x_2 + 5x_3 \\ 4x_1 + 6x_2 - x_3 \\ -x_1 + 3x_2 - 8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 4x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 6x_2 \\ 3x_2 \end{bmatrix} + \begin{bmatrix} 5x_3 \\ -x_3 \\ -8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Usually, the intermediate calculations are not displayed.

Note: The *Study Guide* says, "Check with your instructor whether you need to "show work" on a problem such as Exercise 9."

10. $\begin{array}{rcrcrcr} 4x_1 & + & x_2 & + & 3x_3 & = & 9 \\ x_1 & - & 7x_2 & - & 2x_3 & = & 2 \\ 8x_1 & + & 6x_2 & - & 5x_3 & = & 15 \end{array}$, $\begin{bmatrix} 4x_1+x_2+3x_3 \\ x_1-7x_2-2x_3 \\ 8x_1+6x_2-5x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$

$\begin{bmatrix} 4x_1 \\ x_1 \\ 8x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ -7x_2 \\ 6x_2 \end{bmatrix} + \begin{bmatrix} 3x_3 \\ -2x_3 \\ -5x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$, $x_1\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$

Usually, the intermediate calculations are not displayed.

11. The question

Is $\mathbf{b}$ a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$?

is equivalent to the question

Does the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ have a solution?

The equation

$$x_1\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \qquad (*)$$

$\uparrow$ $\mathbf{a}_1$ $\uparrow$ $\mathbf{a}_2$ $\uparrow$ $\mathbf{a}_3$ $\uparrow$ $\mathbf{b}$

has the same solution set as the linear system whose augmented matrix is

$$M = \begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 5 & 2 \\ 0 & \textcircled{1} & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The linear system corresponding to M *has* a solution, so the vector equation (*) has a solution, and therefore $\mathbf{b}$ *is* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

12. The equation

$$x_1\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix} \qquad (*)$$

$\uparrow$ $\mathbf{a}_1$ $\uparrow$ $\mathbf{a}_2$ $\uparrow$ $\mathbf{a}_3$ $\uparrow$ $\mathbf{b}$

has the same solution set as the linear system whose augmented matrix is

$$M = \begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M \sim \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 2 & -5 \\ 0 & \textcircled{5} & 4 & 1 \\ 0 & 0 & 0 & \textcircled{2} \end{bmatrix}$$

The linear system corresponding to M has *no* solution, so the vector equation (*) has no solution, and therefore $\mathbf{b}$ is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

13. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. To determine if $\mathbf{b}$ is a linear combination of these columns, use the boxed fact on page 34. Row reduced the augmented matrix until you reach echelon form:

$$\begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -4 & 2 & 3 \\ 0 & \textcircled{3} & 5 & -7 \\ 0 & 0 & 0 & \textcircled{3} \end{bmatrix}$$

The system for this augmented matrix is inconsistent, so $\mathbf{b}$ is *not* a linear combination of the columns of A.

14. $[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3 \;\; \mathbf{b}] = \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & -6 & 11 \\ 0 & \textcircled{3} & 7 & -5 \\ 0 & 0 & \textcircled{11} & -2 \end{bmatrix}$. The linear system corresponding to this matrix *has* a solution, so $\mathbf{b}$ is a linear combination of the columns of A.

15. Noninteger weights are acceptable, of course, but some simple choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}, \quad 0\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

$$1\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \quad 1\cdot\mathbf{v}_1 - 1\cdot\mathbf{v}_2 = \begin{bmatrix} 12 \\ -2 \\ -6 \end{bmatrix}$$

16. Some likely choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \quad 0\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

$$1\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \quad 1\cdot\mathbf{v}_1 - 1\cdot\mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$$

17. $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{b}] = \begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & h+8 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 4 \\ 0 & \textcircled{1} & -3 \\ 0 & 0 & h+17 \end{bmatrix}$. The vector **b** is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$ when $h + 17$ is zero, that is, when $h = -17$.

18. $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{y}] = \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -3+2h \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & h \\ 0 & \textcircled{1} & -5 \\ 0 & 0 & 7+2h \end{bmatrix}$. The vector **y** is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ when $7 + 2h$ is zero, that is, when $h = -7/2$.

19. By inspection, $\mathbf{v}_2 = (3/2)\mathbf{v}_1$. Any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is actually just a multiple of $\mathbf{v}_1$. For instance,

$$a\mathbf{v}_1 + b\mathbf{v}_2 = a\mathbf{v}_1 + b(3/2)\mathbf{v}_2 = (a + 3b/2)\mathbf{v}_1$$

So Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is the set of points on the line through $\mathbf{v}_1$ and **0**.

Note: Exercises 19 and 20 prepare the way for ideas in Sections 1.4 and 1.7.

20. Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane in $\mathbf{R}^3$ through the origin, because the neither vector in this problem is a multiple of the other. Every vector in the set has 0 as its second entry and so lies in the xz-plane in ordinary 3-space. So Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ *is* the xz-plane.

21. Let $\mathbf{y} = \begin{bmatrix} h \\ k \end{bmatrix}$. Then $[\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{y}] = \begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & 2 & h \\ 0 & \textcircled{2} & k+h/2 \end{bmatrix}$. This augmented matrix corresponds to a consistent system for all h and k. So **y** is in Span$\{\mathbf{u}, \mathbf{v}\}$ for all h and k.

22. Construct any 3×4 matrix in echelon form that corresponds to an inconsistent system. Perform sufficient row operations on the matrix to eliminate all zero entries in the first three columns.

23. **a**. False. The alternative notation for a (column) vector is (–4, 3), using parentheses and commas.

    b. False. Plot the points to verify this. Or, see the statement preceding Example 3. If $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ were on the line through $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and the origin, then $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ would have to be a multiple of $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$, which is not the case.

    c. True. See the line displayed just before Example 4.

    d. True. See the box that discusses the matrix in (5).

    e. False. The statement is often true, but Span$\{\mathbf{u}, \mathbf{v}\}$ is not a plane when **v** is a multiple of **u**, or when **u** is the zero vector.

24. **a**. True. See the beginning of the subsection *Vectors in* $\mathbf{R}^n$.

    b. True. Use Fig. 7 to draw the parallelogram determined by **u** – **v** and **v**.

    c. False. See the first paragraph of the subsection *Linear Combinations*.

    d. True. See the statement that refers to Fig. 11.

    e. True. See the paragraph following the definition of Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

25. **a.** There are only three vectors in the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, and **b** is not one of them.

b. There are infinitely many vectors in $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. To determine if **b** is in W, use the method of Exercise 13.

$$\begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -4 & 4 \\ 0 & \textcircled{3} & -2 & 1 \\ 0 & 0 & \textcircled{-1} & 2 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{matrix}$$

The system for this augmented matrix is consistent, so **b** is in W.

c. $\mathbf{a}_1 = 1\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3$. See the discussion in the text following the definition of $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

26. **a.** $$[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}] = \begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Yes, **b** is a linear combination of the columns of A, that is, **b** is in W.

b. The third column of A is in W because $\mathbf{a}_3 = 0\cdot\mathbf{a}_1 + 0\cdot\mathbf{a}_2 + 1\cdot\mathbf{a}_3$.

27. **a.** $5\mathbf{v}_1$ is the output of 5 days' operation of mine #1.

b. The total output is $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$, so x_1 and x_2 should satisfy $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} 150 \\ 2825 \end{bmatrix}$.

c. **[M]** Reduce the augmented matrix $\begin{bmatrix} 20 & 30 & 150 \\ 550 & 500 & 2825 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & 4.0 \end{bmatrix}$.

Operate mine #1 for 1.5 days and mine #2 for 4 days. (This is the exact solution.)

28. **a.** The amount of heat produced when the steam plant burns x_1 tons of anthracite and x_2 tons of bituminous coal is $27.6x_1 + 30.2x_2$ million Btu.

b. The total output produced by x_1 tons of anthracite and x_2 tons of bituminous coal is given by the vector $x_1\begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix} + x_2\begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix}$.

c. **[M]** The appropriate values for x_1 and x_2 satisfy $x_1\begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix} + x_2\begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix} = \begin{bmatrix} 162 \\ 23{,}610 \\ 1{,}623 \end{bmatrix}$.

To solve, row reduce the augmented matrix:

$$\begin{bmatrix} 27.6 & 30.2 & 162 \\ 3100 & 6400 & 23610 \\ 250 & 360 & 1623 \end{bmatrix} \sim \begin{bmatrix} 1.000 & 0 & 3.900 \\ 0 & 1.000 & 1.800 \\ 0 & 0 & 0 \end{bmatrix}$$

The steam plant burned 3.9 tons of anthracite coal and 1.8 tons of bituminous coal.

29. The total mass is 2 + 5 + 2 + 1 = 10. So $\mathbf{v} = (2\mathbf{v}_1 + 5\mathbf{v}_2 + 2\mathbf{v}_3 + \mathbf{v}_4)/10$. That is,

$$\mathbf{v} = \frac{1}{10}\left(2\begin{bmatrix}5\\-4\\3\end{bmatrix} + 5\begin{bmatrix}4\\3\\-2\end{bmatrix} + 2\begin{bmatrix}-4\\-3\\-1\end{bmatrix} + \begin{bmatrix}-9\\8\\6\end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix}10+20-8-9\\-8+15-6+8\\6-10-2+6\end{bmatrix} = \begin{bmatrix}1.3\\.9\\0\end{bmatrix}$$

30. Let m be the total mass of the system. By definition,

$$\mathbf{v} = \frac{1}{m}(m_1\mathbf{v}_1 + \cdots + m_k\mathbf{v}_k) = \frac{m_1}{m}\mathbf{v}_1 + \cdots + \frac{m_k}{m}\mathbf{v}_k$$

The second expression displays $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, which shows that $\mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

31. a. The center of mass is $\frac{1}{3}\left(1\cdot\begin{bmatrix}0\\1\end{bmatrix} + 1\cdot\begin{bmatrix}8\\1\end{bmatrix} + 1\cdot\begin{bmatrix}2\\4\end{bmatrix}\right) = \begin{bmatrix}10/3\\2\end{bmatrix}$.

b. The total mass of the new system is 9 grams. The three masses added, w_1, w_2, and w_3, satisfy the equation

$$\frac{1}{9}\left((w_1+1)\cdot\begin{bmatrix}0\\1\end{bmatrix} + (w_2+1)\cdot\begin{bmatrix}8\\1\end{bmatrix} + (w_3+1)\cdot\begin{bmatrix}2\\4\end{bmatrix}\right) = \begin{bmatrix}2\\2\end{bmatrix}$$

which can be rearranged to

$$(w_1+1)\cdot\begin{bmatrix}0\\1\end{bmatrix} + (w_2+1)\cdot\begin{bmatrix}8\\1\end{bmatrix} + (w_3+1)\cdot\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}18\\18\end{bmatrix}$$

and

$$w_1\cdot\begin{bmatrix}0\\1\end{bmatrix} + w_2\cdot\begin{bmatrix}8\\1\end{bmatrix} + w_3\cdot\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}8\\12\end{bmatrix}$$

The condition $w_1 + w_2 + w_3 = 6$ and the vector equation above combine to produce a system of three equations whose augmented matrix is shown below, along with a sequence of row operations:

$$\begin{bmatrix}1&1&1&6\\0&8&2&8\\1&1&4&12\end{bmatrix} \sim \begin{bmatrix}1&1&1&6\\0&8&2&8\\0&0&3&6\end{bmatrix} \sim \begin{bmatrix}1&1&1&6\\0&8&2&8\\0&0&1&2\end{bmatrix}$$

$$\sim \begin{bmatrix}1&1&0&4\\0&8&0&4\\0&0&1&2\end{bmatrix} \sim \begin{bmatrix}1&0&0&3.5\\0&8&0&4\\0&0&1&2\end{bmatrix} \sim \begin{bmatrix}1&0&0&3.5\\0&1&0&.5\\0&0&1&2\end{bmatrix}$$

Answer: Add 3.5 g at (0, 1), add .5 g at (8, 1), and add 2 g at (2, 4).

Extra problem: Ignore the mass of the plate, and distribute 6 gm at the three vertices to make the center of mass at (2, 2). Answer: Place 3 g at (0, 1), 1 g at (8, 1), and 2 g at (2, 4).

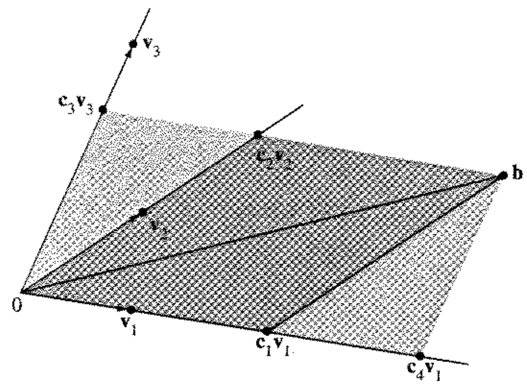
32. See the parallelograms drawn on Fig. 15 from the text. Here c_1, c_2, c_3, and c_4 are suitable scalars. The darker parallelogram shows that $\mathbf{b}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, that is

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\cdot\mathbf{v}_3 = \mathbf{b}$$

The larger parallelogram shows that **b** is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$, that is,

$$c_4\mathbf{v}_1 + 0\cdot\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$$

So the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ has at least two solutions, not just one solution. (In fact, the equation has infinitely many solutions.)



33. **a.** For $j = 1,\ldots, n$, the jth entry of $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$ is $(u_j + v_j) + w_j$. By associativity of addition in **R**, this entry equals $u_j + (v_j + w_j)$, which is the jth entry of $\mathbf{u} + (\mathbf{v} + \mathbf{w})$. By definition of equality of vectors, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

b. For any scalar c, the jth entry of $c(\mathbf{u} + \mathbf{v})$ is $c(u_j + v_j)$, and the jth entry of $c\mathbf{u} + c\mathbf{v}$ is $cu_j + cv_j$ (by definition of scalar multiplication and vector addition). These entries are equal, by a distributive law in **R**. So $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

34. **a.** For $j = 1,\ldots, n$, $u_j + (-1)u_j = (-1)u_j + u_j = 0$, by properties of **R**. By vector equality,

$$\mathbf{u} + (-1)\mathbf{u} = (-1)\mathbf{u} + \mathbf{u} = \mathbf{0}.$$

b. For scalars c and d, the jth entries of $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are $c(du_j)$ and $(cd)u_j$, respectively. These entries in **R** are equal, so the vectors $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are equal.

Note: When an exercise in this section involves a vector equation, the corresponding technology data (in the data files on the web) is usually presented as a set of (column) vectors. To use MATLAB or other technology, a student must first construct an augmented matrix from these vectors. The MATLAB note in the *Study Guide* describes how to do this. The appendices in the *Study Guide* give corresponding information about Maple, Mathematica, and the TI and HP calculators.