1.4 SOLUTIONS

Notes: Key exercises are 1–20, 27, 28, 31 and 32. Exercises 29, 30, 33, and 34 are harder. Exercise 34 anticipates the Invertible Matrix Theorem but is not used in the proof of that theorem.

1. The matrix-vector product $A\mathbf{x}$ product is not defined because the number of columns (2) in the 3×2 matrix $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix}$ does not match the number of entries (3) in the vector $\begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$.

2. The matrix-vector product $A\mathbf{x}$ product is not defined because the number of columns (1) in the 3×1 matrix $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$ does not match the number of entries (2) in the vector $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

3. $A\mathbf{x} = \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2\begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} - 3\begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \\ 14 \end{bmatrix} + \begin{bmatrix} -15 \\ 9 \\ -18 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$, and

$$A\mathbf{x} = \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 6\cdot 2 + 5\cdot(-3) \\ (-4)\cdot 2 + (-3)\cdot(-3) \\ 7\cdot 2 + 6\cdot(-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

4. $A\mathbf{x} = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1\cdot\begin{bmatrix} 8 \\ 5 \end{bmatrix} + 1\cdot\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1\cdot\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8+3-4 \\ 5+1+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$, and

$$A\mathbf{x} = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8\cdot 1 + 3\cdot 1 + (-4)\cdot 1 \\ 5\cdot 1 + 1\cdot 1 + 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

5. On the left side of the matrix equation, use the entries in the vector $\mathbf{x}$ as the weights in a linear combination of the columns of the matrix A:

$$5\cdot\begin{bmatrix} 5 \\ -2 \end{bmatrix} - 1\cdot\begin{bmatrix} 1 \\ -7 \end{bmatrix} + 3\cdot\begin{bmatrix} -8 \\ 3 \end{bmatrix} - 2\cdot\begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$$

6. On the left side of the matrix equation, use the entries in the vector $\mathbf{x}$ as the weights in a linear combination of the columns of the matrix A:

$$-2\cdot\begin{bmatrix} 7 \\ 2 \\ 9 \\ -3 \end{bmatrix} - 5\cdot\begin{bmatrix} -3 \\ 1 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$$

7. The left side of the equation is a linear combination of three vectors. Write the matrix A whose columns are those three vectors, and create a variable vector $\mathbf{x}$ with three entries:

$A = \left[\begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Thus the equation $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

For your information: The unique solution of this equation is (5, 7, 3). Finding the solution by hand would be time-consuming.

Note: The skill of writing a vector equation as a matrix equation will be important for both theory and application throughout the text. See also Exercises 27 and 28.

8. The left side of the equation is a linear combination of four vectors. Write the matrix A whose columns are those four vectors, and create a variable vector with four entries:

$$A = \begin{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} & \begin{bmatrix} -4 \\ 5 \end{bmatrix} & \begin{bmatrix} -5 \\ 4 \end{bmatrix} & \begin{bmatrix} 3 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 4 & -4 & -5 & 3 \\ -2 & 5 & 4 & 0 \end{bmatrix}, \text{ and } \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}. \text{ Then the equation } A\mathbf{z} = \mathbf{b}$$

is $\begin{bmatrix} 4 & -4 & -5 & 3 \\ -2 & 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$.

For your information: One solution is (7, 3, 3, 1). The general solution is $z_1 = 6 + .75z_3 - 1.25z_4$, $z_2 = 5 - .5z_3 - .5z_4$, with z_3 and z_4 free.

9. The system has the same solution set as the vector equation

$$x_1 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix} 3 & 1 & -5 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

10. The system has the same solution set as the vector equation

$$x_1 \begin{bmatrix} 8 \\ 5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix} 8 & -1 \\ 5 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

11. To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3 \ \ \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & -3 \\ 0 & 0 & \textcircled{1} & 1 \end{bmatrix}$$

The solution is $\begin{cases} x_1 &= 0 \\ x_2 &= -3 \\ x_3 &= 1 \end{cases}$. As a vector, the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$.

12. To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3 \ \ \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 3/5 \\ 0 & \textcircled{1} & 0 & -4/5 \\ 0 & 0 & \textcircled{1} & 1 \end{bmatrix}$$

The solution is $\begin{cases} x_1 &= 3/5 \\ x_2 &= -4/5 \\ x_3 &= 1 \end{cases}$. As a vector, the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}$.

13. The vector $\mathbf{u}$ is in the plane spanned by the columns of A if and only if $\mathbf{u}$ is a linear combination of the columns of A. This happens if and only if the equation $A\mathbf{x} = \mathbf{u}$ has a solution. (See the box preceding Example 3 in Section 1.4.) To study this equation, reduce the augmented matrix $[A \ \ \mathbf{u}]$

$$\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ -2 & 6 & 4 \\ 3 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & -8 & -12 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 1 & 4 \\ 0 & \textcircled{8} & 12 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{u}$ has a solution, so $\mathbf{u}$ is in the plane spanned by the columns of A.

For your information: The unique solution of $A\mathbf{x} = \mathbf{u}$ is (5/2, 3/2).

14. Reduce the augmented matrix $[A \ \ \mathbf{u}]$ to echelon form:

$$\begin{bmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & 0 & 2 \\ 0 & \textcircled{1} & -1 & -3 \\ 0 & 0 & 0 & \textcircled{-29} \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{u}$ has no solution, so $\mathbf{u}$ is not in the subset spanned by the columns of A.

15. The augmented matrix for $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 2 & -1 & b_1 \\ -6 & 3 & b_2 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} \textcircled{2} & -1 & b_1 \\ 0 & 0 & b_2 + 3b_1 \end{bmatrix}$.

This shows that the equation $A\mathbf{x} = \mathbf{b}$ is not consistent when $3b_1 + b_2$ is nonzero. The set of $\mathbf{b}$ for which the equation *is* consistent is a line through the origin–the set of all points (b_1, b_2) satisfying $b_2 = -3b_1$.

16. Row reduce the augmented matrix $[A \ \ \mathbf{b}]$: $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 14 & 12 & b_3 - 5b_1 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2+3b_1 \\ 0 & 0 & 0 & b_3-5b_1+2(b_2+3b_1) \end{bmatrix} = \begin{bmatrix} \textcircled{1} & -3 & -4 & b_1 \\ 0 & \textcircled{-7} & -6 & b_2+3b_1 \\ 0 & 0 & 0 & b_1+2b_2+b_3 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $b_1 + 2b_2 + b_3 = 0$. The set of such $\mathbf{b}$ is a plane through the origin in $\mathbf{R}^3$.

17. Row reduction shows that only three rows of A contain a pivot position:

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & 0 & 3 \\ 0 & \textcircled{2} & -1 & 4 \\ 0 & 0 & 0 & \textcircled{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because not every row of A contains a pivot position, Theorem 4 in Section 1.4 shows that the equation $A\mathbf{x} = \mathbf{b}$ does *not* have a solution for each $\mathbf{b}$ in $\mathbf{R}^4$.

18. Row reduction shows that only three rows of B contain a pivot position:

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & 5 \\ 0 & -2 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & -2 & 2 \\ 0 & \textcircled{1} & 1 & -5 \\ 0 & 0 & 0 & \textcircled{-7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because not every row of B contains a pivot position, Theorem 4 in Section 1.4 shows that the equation $B\mathbf{x} = \mathbf{y}$ does *not* have a solution for each $\mathbf{y}$ in $\mathbf{R}^4$.

19. The work in Exercise 17 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, not all vectors in $\mathbf{R}^4$ can be written as a linear combination of the columns of A. Also, the columns of A do *not* span $\mathbf{R}^4$.

20. The work in Exercise 18 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, not all vectors in $\mathbf{R}^4$ can be written as a linear combination of the columns of B. The columns of B certainly do *not* span $\mathbf{R}^3$, because each column of B is in $\mathbf{R}^4$, not $\mathbf{R}^3$. (This question was asked to alert students to a fairly common misconception among students who are just learning about spanning.)

21. Row reduce the matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 1 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$ does not have a pivot in each row, so the columns of the matrix do not span $\mathbf{R}^4$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span $\mathbf{R}^4$.

Note: Some students may realize that row operations are not needed, and thereby discover the principle covered in Exercises 31 and 32.

22. Row reduce the matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-2} & 8 & -5 \\ 0 & \textcircled{-3} & -1 \\ 0 & 0 & \textcircled{4} \end{bmatrix}$$

The matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$ has a pivot in each row, so the columns of the matrix span $\mathbf{R}^4$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbf{R}^4$.

23. **a.** False. See the paragraph following equation (3). The text calls $A\mathbf{x} = \mathbf{b}$ a *matrix equation*.
 b. True. See the box before Example 3.
 c. False. See the warning following Theorem 4.
 d. True. See Example 4.
 e. True. See parts (c) and (a) in Theorem 4.
 f. True. In Theorem 4, statement (a) is false if and only if statement (d) is also false.

24. **a.** True. This statement is in Theorem 3. However, the statement is true without any "proof" because, by definition, $A\mathbf{x}$ is simply a notation for $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$, where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the columns of A.
 b. True. See Example 2.
 c. True, by Theorem 3.
 d. True. See the box before Example 2. Saying that **b** is not in the set spanned by the columns of A is the same a saying that **b** is not a linear combination of the columns of A.
 e. False. See the warning that follows Theorem 4.
 f. True. In Theorem 4, statement (c) is false if and only if statement (a) is also false.

25. By definition, the matrix-vector product on the left is a linear combination of the columns of the matrix, in this case using weights –3, –1, and 2. So $c_1 = -3$, $c_2 = -1$, and $c_3 = 2$.

26. The equation in x_1 and x_2 involves the vectors **u**, **v**, and **w**, and it may be viewed as

$[\mathbf{u} \;\; \mathbf{v}]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{w}$. By definition of a matrix-vector product, $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w}$. The stated fact that $3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$ can be rewritten as $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$. So, a solution is $x_1 = 3$, $x_2 = -5$.

27. Place the vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ into the columns of a matrix, say, Q and place the weights x_1, x_2, and x_3 into a vector, say, **x**. Then the vector equation becomes

$$Q\mathbf{x} = \mathbf{v}, \text{ where } Q = [\mathbf{q}_1 \;\; \mathbf{q}_2 \;\; \mathbf{q}_3] \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note: If your answer is the equation $A\mathbf{x} = \mathbf{b}$, you need to specify what A and **b** are.

28. The matrix equation can be written as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{v}_6$, where $c_1 = -3$, $c_2 = 2$, $c_3 = 4$, $c_4 = -1$, $c_5 = 2$, and

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 9 \\ -2 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 7 \\ -4 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

29. Start with any 3×3 matrix B in echelon form that has three pivot positions. Perform a row operation (a row interchange or a row replacement) that creates a matrix A that is *not* in echelon form. Then A has the desired property. The justification is given by row reducing A to B, in order to display the pivot positions. Since A has a pivot position in every row, the columns of A span $\mathbf{R}^3$, by Theorem 4.

30. Start with any nonzero 3×3 matrix B in echelon form that has fewer than three pivot positions. Perform a row operation that creates a matrix A that is *not* in echelon form. Then A has the desired property. Since A does not have a pivot position in every row, the columns of A do not span $\mathbf{R}^3$, by Theorem 4.

31. A 3×2 matrix has three rows and two columns. With only two columns, A can have at most two pivot columns, and so A has at most two pivot positions, which is not enough to fill all three rows. By Theorem 4, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbf{R}^3$. Generally, if A is an m×n matrix with $m > n$, then A can have at most n pivot positions, which is not enough to fill all m rows. Thus, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbf{R}^3$.

32. A set of three vectors in cannot span $\mathbf{R}^4$. Reason: the matrix A whose columns are these three vectors has four rows. To have a pivot in each row, A would have to have at least four columns (one for each pivot), which is not the case. Since A does not have a pivot in every row, its columns do not span $\mathbf{R}^4$, by Theorem 4. In general, a set of n vectors in $\mathbf{R}^m$ cannot span $\mathbf{R}^m$ when n is less than m.

33. If the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the associated system of equations does not have any free variables. If every variable is a basic variable, then each column of A is a pivot column. So the reduced echelon form of A must be $\begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \end{bmatrix}$.

Note: Exercises 33 and 34 are difficult in the context of this section because the focus in Section 1.4 is on existence of solutions, not uniqueness. However, these exercises serve to review ideas from Section 1.2, and they anticipate ideas that will come later.

34. If the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the associated system of equations does not have any free variables. If every variable is a basic variable, then each column of A is a pivot column. So the reduced echelon form of A must be $\begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$. Now it is clear that A has a pivot position in each *row*.

By Theorem 4, the columns of A span $\mathbf{R}^3$.

35. Given $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$, you are asked to show that the equation $A\mathbf{x} = \mathbf{w}$ has a solution, where $\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2$. Observe that $\mathbf{w} = A\mathbf{x}_1 + A\mathbf{x}_2$ and use Theorem 5(a) with $\mathbf{x}_1$ and $\mathbf{x}_2$ in place of $\mathbf{u}$ and $\mathbf{v}$, respectively. That is, $\mathbf{w} = A\mathbf{x}_1 + A\mathbf{x}_2 = A(\mathbf{x}_1 + \mathbf{x}_2)$. So the vector $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ is a solution of $\mathbf{w} = A\mathbf{x}$.

36. Suppose that y and $\mathbf{z}$ satisfy $A\mathbf{y} = \mathbf{z}$. Then $4\mathbf{z} = 4A\mathbf{y}$. By Theorem 5(b), $4A\mathbf{y} = A(4\mathbf{y})$. So $4\mathbf{z} = A(4\mathbf{y})$, which shows that $4\mathbf{y}$ is a solution of $A\mathbf{x} = 4\mathbf{z}$. Thus, the equation $A\mathbf{x} = 4\mathbf{z}$ is consistent.

37. [M] $$\begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix} \sim \begin{bmatrix} 7 & 2 & -5 & 8 \\ 0 & -11/7 & 3/7 & -23/7 \\ 0 & 58/7 & 16/7 & 1/7 \\ 0 & 11 & -3 & 23 \end{bmatrix} \sim \begin{bmatrix} \textcircled{7} & 2 & -5 & 8 \\ 0 & \textcircled{-11/7} & 3/7 & -23/7 \\ 0 & 0 & \textcircled{50/11} & -189/11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

or, approximately $\begin{bmatrix} \textcircled{7} & 2 & -5 & 8 \\ 0 & \textcircled{-1.57} & .429 & -3.29 \\ 0 & 0 & \textcircled{4.55} & -17.2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, to three significant figures. The original matrix does not have a pivot in every row, so its columns do not span $\mathbf{R}^4$, by Theorem 4.

38. **[M]** $\begin{bmatrix} 5 & -7 & -4 & 9 \\ 6 & -8 & -7 & 5 \\ 4 & -4 & -9 & -9 \\ -9 & 11 & 16 & 7 \end{bmatrix} \sim \begin{bmatrix} 5 & -7 & -4 & 9 \\ 0 & 2/5 & -11/5 & -29/5 \\ 0 & 8/5 & -29/5 & -81/5 \\ 0 & -8/5 & 44/5 & 116/5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{5} & -7 & -4 & 9 \\ 0 & \textcircled{2/5} & -11/5 & -29/5 \\ 0 & 0 & \textcircled{3} & 7 \\ 0 & 0 & * & * \end{bmatrix}$

MATLAB shows starred entries for numbers that are essentially zero (to many decimal places). So, with pivots only in the first three rows, the original matrix has columns that do not span $\mathbf{R}^4$, by Theorem 4.

39. **[M]** $\begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ -9 & 4 & -8 & 7 & -3 \\ -6 & 11 & -7 & 3 & -9 \\ 4 & -6 & 10 & -5 & 12 \end{bmatrix} \sim \begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ 0 & -5/4 & 1/4 & 1/4 & 3/4 \\ 0 & 15/2 & -3/2 & -3/2 & -13/2 \\ 0 & -11/3 & 19/3 & -2 & 31/3 \end{bmatrix}$

$\sim \begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ 0 & -5/4 & 1/4 & 1/4 & 3/4 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 28/5 & -41/15 & 122/15 \end{bmatrix} \sim \begin{bmatrix} \textcircled{12} & -7 & 11 & -9 & 5 \\ 0 & \textcircled{-5/4} & 1/4 & 1/4 & 3/4 \\ 0 & 0 & \textcircled{28/5} & -41/15 & 122/15 \\ 0 & 0 & 0 & 0 & \textcircled{-2} \end{bmatrix}$

The original matrix has a pivot in every row, so its columns span $\mathbf{R}^4$, by Theorem 4.

40. **[M]** $\begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ -7 & -8 & 5 & 6 & -9 \\ 11 & 7 & -7 & -9 & -6 \\ -3 & 4 & 1 & 8 & 7 \end{bmatrix} \sim \begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ 0 & 13/8 & -1/4 & -1/8 & 19/8 \\ 0 & -65/8 & 5/4 & 5/8 & -191/8 \\ 0 & 65/8 & -5/4 & 43/8 & 95/8 \end{bmatrix}$

$\sim \begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ 0 & 13/8 & -1/4 & -1/8 & 19/8 \\ 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{8} & 11 & -6 & -7 & 13 \\ 0 & \textcircled{13/8} & -1/4 & -1/8 & 19/8 \\ 0 & 0 & 0 & \textcircled{6} & 0 \\ 0 & 0 & 0 & 0 & \textcircled{-12} \end{bmatrix}$

The original matrix has a pivot in every row, so its columns span $\mathbf{R}^4$, by Theorem 4.

41. **[M]** Examine the calculations in Exercise 39. Notice that the fourth column of the original matrix, say A, is not a pivot column. Let A^o be the matrix formed by deleting column 4 of A, let B be the echelon form obtained from A, and let B^o be the matrix obtained by deleting column 4 of B. The sequence of row operations that reduces A to B also reduces A^o to B^o. Since B^o is in echelon form, it shows that A^o has a pivot position in each row. Therefore, the columns of A^o span $\mathbf{R}^4$.

It is possible to delete column 3 of A instead of column 4. In this case, the fourth column of A becomes a pivot column of A^o, as you can see by looking at what happens when column 3 of B is deleted. For later work, it is desirable to delete a nonpivot column.

Note: Exercises 41 and 42 help to prepare for later work on the column space of a matrix. (See Section 2.9 or 4.6.) The *Study Guide* points out that these exercises depend on the following idea, not explicitly mentioned in the text: when a row operation is performed on a matrix A, the calculations for each new entry depend only on the other entries in the *same column*. If a column of A is removed, forming a new matrix, the absence of this column has no affect on any row-operation calculations for entries in the other columns of A. (The absence of a column might affect the particular *choice* of row operations performed for some purpose, but that is not being considered here.)

42. **[M]** Examine the calculations in Exercise 40. The third column of the original matrix, say A, is not a pivot column. Let A^o be the matrix formed by deleting column 3 of A, let B be the echelon form obtained from A, and let B^o be the matrix obtained by deleting column 3 of B. The sequence of row operations that reduces A to B also reduces A^o to B^o. Since B^o is in echelon form, it shows that A^o has a pivot position in each row. Therefore, the columns of A^o span $\mathbf{R}^4$.

It is possible to delete column 2 of A instead of column 3. (See the remark for Exercise 41.) However, only *one* column can be deleted. If two or more columns were deleted from A, the resulting matrix would have fewer than four columns, so it would have fewer than four pivot positions. In such a case, not every row could contain a pivot position, and the columns of the matrix would not span $\mathbf{R}^4$, by Theorem 4.

Notes: At the end of Section 1.4, the *Study Guide* gives students a method for learning and mastering linear algebra concepts. Specific directions are given for constructing a review sheet that connects the basic definition of "span" with related ideas: equivalent descriptions, theorems, geometric interpretations, special cases, algorithms, and typical computations. I require my students to prepare such a sheet that reflects their choices of material connected with "span", and I make comments on their sheets to help them refine their review. Later, the students use these sheets when studying for exams.

The MATLAB box for Section 1.4 introduces two useful commands **`gauss`** and **`bgauss`** that allow a student to speed up row reduction while still visualizing all the steps involved. The command **`B = gauss(A,1)`** causes MATLAB to find the left-most nonzero entry in row 1 of matrix A, and use that entry as a pivot to create zeros in the entries below, using row replacement operations. The result is a matrix that a student might write next to A as the first stage of row reduction, since there is no need to write a new matrix after each separate row replacement. I use the **`gauss`** command frequently in lectures to obtain an echelon form that provides data for solving various problems. For instance, if a matrix has 5 rows, and if row swaps are not needed, the following commands produce an echelon form of A:

```
B = gauss(A,1),  B = gauss(B,2),  B = gauss(B,3),  B = gauss(B,4)
```

If an interchange is required, I can insert a command such as **`B = swap(B,2,5)`** . The command **`bgauss`** uses the left-most nonzero entry in a row to produce zeros *above* that entry. This command, together with **`scale`**, can change an echelon form into reduced echelon form.

The use of **`gauss`** and **`bgauss`** creates an environment in which students use their computer program the same way they work a problem by hand on an exam. Unless you are able to conduct your exams in a computer laboratory, it may be unwise to give students too early the power to obtain reduced echelon forms with one command—they may have difficulty performing row reduction by hand during an exam. Instructors whose students use a graphic calculator in class each day do not face this problem. In such a case, you may wish to introduce **`rref`** earlier in the course than Chapter 4 (or Section 2.8), which is where I finally allow students to use that command.