1.5 SOLUTIONS

Notes: The geometry helps students understand Span$\{\mathbf{u}, \mathbf{v}\}$, in preparation for later discussions of subspaces. The parametric vector form of a solution set will be used throughout the text. Figure 6 will appear again in Sections 2.9 and 4.8.

For solving homogeneous systems, the text recommends working with the augmented matrix, although no calculations take place in the augmented column. See the *Study Guide* comments on Exercise 7 that illustrate two common student errors.

All students need the practice of Exercises 1–14. (Assign all odd, all even, or a mixture. If you do not assign Exercise 7, be sure to assign both 8 and 10.) Otherwise, a few students may be unable later to find a basis for a null space or an eigenspace. Exercises 29–34 are important. Exercises 33 and 34 help students later understand how solutions of $A\mathbf{x} = \mathbf{0}$ encode linear-dependence relations among the columns of A. Exercises 35–38 are more challenging. Exercise 37 will help students avoid the standard mistake of forgetting that Theorem 6 applies only to a *consistent* equation $A\mathbf{x} = \mathbf{b}$.

1. Reduce the augmented matrix to echelon form and circle the pivot positions. If a column of the *coefficient* matrix is not a pivot column, the corresponding variable is free and the system of equations has a nontrivial solution. Otherwise, the system has *only* the trivial solution.

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variable x_3 is free, so the system has a nontrivial solution.

2. $$\begin{bmatrix} 1 & -3 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 2 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 0 & 12 & 0 \end{bmatrix}$$

There is no free variable; the system has only the trivial solution.

3. $\begin{bmatrix} -3 & 5 & -7 & 0 \\ -6 & 7 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & 5 & -7 & 0 \\ 0 & -3 & 15 & 0 \end{bmatrix}$. The variable x_3 is free; the system has nontrivial solutions.

An alert student will realize that row operations are unnecessary. With only two equations, there can be at most two basic variables. One variable *must* be free. Refer to Exercise 31 in Section 1.2.

4. $\begin{bmatrix} -5 & 7 & 9 & 0 \\ 1 & -2 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 6 & 0 \\ -5 & 7 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 6 & 0 \\ 0 & -3 & 39 & 0 \end{bmatrix}$. x_3 is a free variable; the system has nontrivial solutions. As in Exercise 3, row operations are unnecessary.

5. $$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad - 5x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

The variable x_3 is free, $x_1 = 5x_3$, and $x_2 = -2x_3$.

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{aligned} x_1 \quad + 4x_3 &= 0 \\ x_2 - 3x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

. The variable x_3 is free, $x_1 = -4x_3$, and $x_2 = 3x_3$.

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & -8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$. $\quad \begin{aligned} x_1 \quad + 9x_3 - 8x_4 &= 0 \\ x_2 - 4x_3 + 5x_4 &= 0 \end{aligned}$

The basic variables are x_1 and x_2, with x_3 and x_4 free. Next, $x_1 = -9x_3 + 8x_4$, and $x_2 = 4x_3 - 5x_4$. The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8x_4 \\ -5x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

8. $\begin{bmatrix} 1 & -2 & -9 & 5 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & -7 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix}$. $\quad \begin{aligned} x_1 \quad - 5x_3 - 7x_4 &= 0 \\ x_2 + 2x_3 - 6x_4 &= 0 \end{aligned}$

The basic variables are x_1 and x_2, with x_3 and x_4 free. Next, $x_1 = 5x_3 + 7x_4$ and $x_2 = -2x_3 + 6x_4$. The general solution in parametric vector form is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 + 7x_4 \\ -2x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 6x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

9. $\begin{bmatrix} 3 & -9 & 6 & 0 \\ -1 & 3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 3 & -9 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ 0 &= 0 \end{aligned}$

The solution is $x_1 = 3x_2 - 2x_3$, with x_2 and x_3 free. In parametric vector form,

$$\mathbf{x} = \begin{bmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

10. $\begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 - 3x_2 \quad - 4x_4 &= 0 \\ 0 &= 0 \end{aligned}$

The only basic variable is x_1, so x_2, x_3, and x_4 are free. (Note that x_3 is not zero.) Also, $x_1 = 3x_2 + 4x_4$. The general solution is

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}3x_2+4x_4\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}3x_2\\x_2\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\x_3\\0\end{bmatrix}+\begin{bmatrix}4x_4\\0\\0\\x_4\end{bmatrix}=x_2\begin{bmatrix}3\\1\\0\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+x_4\begin{bmatrix}4\\0\\0\\1\end{bmatrix}$$

11. $$\begin{bmatrix}1&-4&-2&0&3&-5&0\\0&0&1&0&0&-1&0\\0&0&0&0&1&-4&0\\0&0&0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&-4&-2&0&0&7&0\\0&0&1&0&0&-1&0\\0&0&0&0&1&-4&0\\0&0&0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&-4&0&0&0&5&0\\0&0&1&0&0&-1&0\\0&0&0&0&1&-4&0\\0&0&0&0&0&0&0\end{bmatrix}$$

$$\begin{aligned}x_1-4x_2\qquad\qquad+5x_6&=0\\x_3\qquad-x_6&=0\\x_5-4x_6&=0\\0&=0\end{aligned}$$

. The basic variables are x_1, x_3, and x_5. The remaining variables are free.

In particular, x_4 is free (and not zero as some may assume). The solution is $x_1=4x_2-5x_6$, $x_3=x_6$, $x_5=4x_6$, with x_2, x_4, and x_6 free. In parametric vector form,

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}4x_2-5x_6\\x_2\\x_6\\x_4\\4x_6\\x_6\end{bmatrix}=\begin{bmatrix}4x_2\\x_2\\0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\x_4\\0\\0\end{bmatrix}+\begin{bmatrix}-5x_6\\0\\x_6\\0\\4x_6\\x_6\end{bmatrix}=x_2\underset{\substack{\uparrow\\ \mathbf{u}}}{\begin{bmatrix}4\\1\\0\\0\\0\\0\end{bmatrix}}+x_4\underset{\substack{\uparrow\\ \mathbf{v}}}{\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}}+x_6\underset{\substack{\uparrow\\ \mathbf{w}}}{\begin{bmatrix}-5\\0\\1\\0\\4\\1\end{bmatrix}}$$

Note: The *Study Guide* discusses two mistakes that students often make on this type of problem.

12. $$\begin{bmatrix}1&5&2&-6&9&0&0\\0&0&1&-7&4&-8&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&5&2&-6&9&0&0\\0&0&1&-7&4&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&5&0&8&1&0&0\\0&0&1&-7&4&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\end{bmatrix}$$

$$\begin{aligned}x_1+5x_2\qquad+8x_4+x_5\qquad&=0\\x_3-7x_4+4x_5\qquad&=0\\x_6&=0\\0&=0\end{aligned}$$

The basic variables are x_1, x_3, and x_6; the free variables are x_2, x_4, and x_5. The general solution is $x_1=-5x_2-8x_4-x_5$, $x_3=7x_4-4x_5$, and $x_6=0$. In parametric vector form, the solution is

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}-5x_2-8x_4-x_5\\x_2\\7x_4-4x_5\\x_4\\x_5\\0\end{bmatrix}=\begin{bmatrix}-5x_2\\x_2\\0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}-8x_4\\0\\7x_4\\x_4\\0\\0\end{bmatrix}+\begin{bmatrix}-x_5\\0\\-4x_5\\0\\x_5\\0\end{bmatrix}=x_2\begin{bmatrix}-5\\1\\0\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-8\\0\\7\\1\\0\\0\end{bmatrix}+x_5\begin{bmatrix}-1\\0\\-4\\0\\1\\0\end{bmatrix}$$

13. To write the general solution in parametric vector form, pull out the constant terms that do not involve the free variable:

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}5+4x_3\\-2-7x_3\\x_3\end{bmatrix}=\begin{bmatrix}5\\-2\\0\end{bmatrix}+\begin{bmatrix}4x_3\\-7x_3\\x_3\end{bmatrix}=\underset{\substack{\uparrow\\ \mathbf{p}}}{\begin{bmatrix}5\\-2\\0\end{bmatrix}}+x_3\underset{\substack{\uparrow\\ \mathbf{q}}}{\begin{bmatrix}4\\-7\\1\end{bmatrix}}=\mathbf{p}+x_3\mathbf{q}.$$

Geometrically, the solution set is the line through $\begin{bmatrix}5\\-2\\0\end{bmatrix}$ in the direction of $\begin{bmatrix}4\\-7\\1\end{bmatrix}$.

14. To write the general solution in parametric vector form, pull out the constant terms that do not involve the free variable:

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}3x_4\\8+x_4\\2-5x_4\\x_4\end{bmatrix}=\begin{bmatrix}0\\8\\2\\0\end{bmatrix}+\begin{bmatrix}3x_4\\x_4\\-5x_4\\x_4\end{bmatrix}=\underset{\substack{\uparrow\\ \mathbf{p}}}{\begin{bmatrix}0\\8\\2\\0\end{bmatrix}}+x_4\underset{\substack{\uparrow\\ \mathbf{q}}}{\begin{bmatrix}3\\1\\-5\\1\end{bmatrix}}=\mathbf{p}+x_4\mathbf{q}$$

The solution set is the line through $\mathbf{p}$ in the direction of $\mathbf{q}$.

15. Row reduce the augmented matrix for the system:

$$\begin{bmatrix}1&3&1&1\\-4&-9&2&-1\\0&-3&-6&-3\end{bmatrix}\sim\begin{bmatrix}1&3&1&1\\0&3&6&3\\0&-3&-6&-3\end{bmatrix}\sim\begin{bmatrix}1&3&1&1\\0&3&6&3\\0&0&0&0\end{bmatrix}$$

$$\sim\begin{bmatrix}1&3&1&1\\0&1&2&1\\0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&-5&-2\\0&1&2&1\\0&0&0&0\end{bmatrix}.\qquad\begin{aligned}x_1\quad-5x_3&=-2\\x_2+2x_3&=1.\\0&=0\end{aligned}$$

Thus $x_1=-2+5x_3$, $x_2=1-2x_3$, and x_3 is free. In parametric vector form,

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-2+5x_3\\1-2x_3\\x_3\end{bmatrix}=\begin{bmatrix}-2\\1\\0\end{bmatrix}+\begin{bmatrix}5x_3\\-2x_3\\x_3\end{bmatrix}=\begin{bmatrix}-2\\1\\0\end{bmatrix}+x_3\begin{bmatrix}5\\-2\\1\end{bmatrix}$$

The solution set is the line through $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, parallel to the line that is the solution set of the homogeneous system in Exercise 5.

16. Row reduce the augmented matrix for the system:

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad + 4x_3 &= -5 \\ x_2 - 3x_3 &= 3 \\ 0 &= 0 \end{aligned}$$

. Thus $x_1 = -5 - 4x_3$, $x_2 = 3 + 3x_3$, and x_3 is free. In parametric vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5-4x_3 \\ 3+3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

The solution set is the line through $\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$, parallel to the line that is the solution set of the homogeneous system in Exercise 6.

17. Solve $x_1 + 9x_2 - 4x_3 = -2$ for the basic variable: $x_1 = -2 - 9x_2 + 4x_3$, with x_2 and x_3 free. In vector form, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2-9x_2+4x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -9x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

The solution of $x_1 + 9x_2 - 4x_3 = 0$ is $x_1 = -9x_2 + 4x_3$, with x_2 and x_3 free. In vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9x_2+4x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_3\mathbf{v}$$

The solution set of the homogeneous equation is the plane through the origin in $\mathbf{R}^3$ spanned by $\mathbf{u}$ and $\mathbf{v}$. The solution set of the nonhomogeneous equation is parallel to this plane and passes through the point $\mathbf{p} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$.

18. Solve $x_1 - 3x_2 + 5x_3 = 4$ for the basic variable: $x_1 = 4 + 3x_2 - 5x_3$, with x_2 and x_3 free. In vector form, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4+3x_2-5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

The solution of $x_1 - 3x_2 + 5x_3 = 0$ is $x_1 = 3x_2 - 5x_3$, with x_2 and x_3 free. In vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_3\mathbf{v}$$

The solution set of the homogeneous equation is the plane through the origin in $\mathbf{R}^3$ spanned by **u** and **v**. The solution set of the nonhomogeneous equation is parallel to this plane and passes through the point $\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$.

19. The line through **a** parallel to **b** can be written as $\mathbf{x} = \mathbf{a} + t\,\mathbf{b}$, where t represents a parameter:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 3 \end{bmatrix}, \text{ or } \begin{cases} x_1 = -2 - 5t \\ x_2 = 3t \end{cases}$$

20. The line through **a** parallel to **b** can be written as $\mathbf{x} = \mathbf{a} + t\mathbf{b}$, where t represents a parameter:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + t\begin{bmatrix} -7 \\ 8 \end{bmatrix}, \text{ or } \begin{cases} x_1 = 3 - 7t \\ x_2 = -4 + 8t \end{cases}$$

21. The line through **p** and **q** is parallel to $\mathbf{q} - \mathbf{p}$. So, given $\mathbf{p} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, form

$$\mathbf{q} - \mathbf{p} = \begin{bmatrix} -3-2 \\ 1-(-5) \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}, \text{ and write the line as } \mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{bmatrix} 2 \\ -5 \end{bmatrix} + t\begin{bmatrix} -5 \\ 6 \end{bmatrix}.$$

22. The line through **p** and **q** is parallel to $\mathbf{q} - \mathbf{p}$. So, given $\mathbf{p} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$, form

$$\mathbf{q} - \mathbf{p} = \begin{bmatrix} 0-(-6) \\ -4-3 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}, \text{ and write the line as } \mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{bmatrix} -6 \\ 3 \end{bmatrix} + t\begin{bmatrix} 6 \\ -7 \end{bmatrix}$$

Note: Exercises 21 and 22 prepare for Exercise 27 in Section 1.8.

23. **a.** True. See the first paragraph of the subsection titled *Homogeneous Linear Systems*.

b. False. The equation $A\mathbf{x} = \mathbf{0}$ gives an *implicit* description of its solution set. See the subsection entitled *Parametric Vector Form*.

c. False. The equation $A\mathbf{x} = \mathbf{0}$ *always* has the trivial solution. The box before Example 1 uses the word *nontrivial* instead of *trivial*.

d. False. The line goes through **p** parallel to **v**. See the paragraph that precedes Fig. 5.

e. False. The solution set could be *empty*! The statement (from Theorem 6) is true only when there exists a vector **p** such that $A\mathbf{p} = \mathbf{b}$.

24. **a.** False. A nontrivial solution of $A\mathbf{x} = \mathbf{0}$ is any nonzero **x** that satisfies the equation. See the sentence before Example 2.

b. True. See Example 2 and the paragraph following it.

c. True. If the zero vector is a solution, then $\mathbf{b} = A\mathbf{x} = A\mathbf{0} = \mathbf{0}$.
d. True. See the paragraph following Example 3.
e. False. The statement is true only when the solution set of $A\mathbf{x} = \mathbf{0}$ is nonempty. Theorem 6 applies only to a consistent system.

25. Suppose $\mathbf{p}$ satisfies $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{p} = \mathbf{b}$. Theorem 6 says that the solution set of $A\mathbf{x} = \mathbf{b}$ equals the set $S = \{\mathbf{w} : \mathbf{w} = \mathbf{p} + \mathbf{v}_h \text{ for some } \mathbf{v}_h \text{ such that } A\mathbf{v}_h = \mathbf{0}\}$. There are two things to prove: (a) every vector in S satisfies $A\mathbf{x} = \mathbf{b}$, (b) every vector that satisfies $A\mathbf{x} = \mathbf{b}$ is in S.

a. Let $\mathbf{w}$ have the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $A\mathbf{v}_h = \mathbf{0}$. Then

$$A\mathbf{w} = A(\mathbf{p} + \mathbf{v}_h) = A\mathbf{p} + A\mathbf{v}_h. \text{ By Theorem 5(a) in section 1.4}$$
$$= \mathbf{b} + \mathbf{0} = \mathbf{b}$$

So every vector of the form $\mathbf{p} + \mathbf{v}_h$ satisfies $A\mathbf{x} = \mathbf{b}$.

b. Now let $\mathbf{w}$ be any solution of $A\mathbf{x} = \mathbf{b}$, and set $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$. Then

$$A\mathbf{v}_h = A(\mathbf{w} - \mathbf{p}) = A\mathbf{w} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

So $\mathbf{v}_h$ satisfies $A\mathbf{x} = \mathbf{0}$. Thus every solution of $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$.

26. (*Geometric argument using Theorem* 6.) Since $A\mathbf{x} = \mathbf{b}$ is consistent, its solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, by Theorem 6. So the solution set of $A\mathbf{x} = \mathbf{b}$ is a single vector if and only if the solution set of $A\mathbf{x} = \mathbf{0}$ is a single vector, and that happens if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*Proof using free variables.*) If $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution is unique if and only if there are no free variables in the corresponding system of equations, that is, if and only if every column of A is a pivot column. This happens if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

27. When A is the 3×3 zero matrix, *every* $\mathbf{x}$ in $\mathbf{R}^3$ satisfies $A\mathbf{x} = \mathbf{0}$. So the solution set is all vectors in $\mathbf{R}^3$.

28. No. If the solution set of $A\mathbf{x} = \mathbf{b}$ contained the origin, then $\mathbf{0}$ would satisfy $A\mathbf{0} = \mathbf{b}$, which is not true since $\mathbf{b}$ is not the zero vector.

29. a. When A is a 3×3 matrix with three pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence has no nontrivial solution.

b. With three pivot positions, A has a pivot position in each of its three rows. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every possible $\mathbf{b}$. The term "possible" in the exercise means that the only vectors considered in this case are those in $\mathbf{R}^3$, because A has three rows.

30. a. When A is a 3×3 matrix with two pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has two basic variables and one free variable. So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

b. With only two pivot positions, A cannot have a pivot in every row, so by Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible $\mathbf{b}$ (in $\mathbf{R}^3$).

31. a. When A is a 3×2 matrix with two pivot positions, each column is a pivot column. So the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence no nontrivial solution.

b. With two pivot positions and three rows, A cannot have a pivot in every row. So the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible $\mathbf{b}$ (in $\mathbf{R}^3$), by Theorem 4 in Section 1.4.

32. a. When A is a 2×4 matrix with two pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has two basic variables and two free variables. So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

b. With two pivot positions and only two rows, A has a pivot position in every row. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every possible $\mathbf{b}$ (in $\mathbf{R}^2$).

33. Look at $x_1\begin{bmatrix}-2\\7\\-3\end{bmatrix}+x_2\begin{bmatrix}-6\\21\\-9\end{bmatrix}$ and notice that the second column is 3 times the first. So suitable values for x_1 and x_2 would be 3 and –1 respectively. (Another pair would be 6 and –2, etc.) Thus $\mathbf{x}=\begin{bmatrix}3\\-1\end{bmatrix}$ satisfies $A\mathbf{x}=\mathbf{0}$.

34. Inspect how the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ of A are related. The second column is –3/2 times the first. Put another way, $3\mathbf{a}_1+2\mathbf{a}_2=\mathbf{0}$. Thus $\begin{bmatrix}3\\2\end{bmatrix}$ satisfies $A\mathbf{x}=\mathbf{0}$.

Note: Exercises 33 and 34 set the stage for the concept of linear dependence.

35. Look for $A=[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3]$ such that $1\cdot\mathbf{a}_1+1\mathbf{a}_2+1\cdot\mathbf{a}_3=\mathbf{0}$. That is, construct A so that each row sum (the sum of the entries in a row) is zero.

36. Look for $A=[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3]$ such that $1\cdot\mathbf{a}_1-2\cdot\mathbf{a}_2+1\cdot\mathbf{a}_3=\mathbf{0}$. That is, construct A so that the sum of the first and third columns is twice the second column.

37. Since the solution set of $A\mathbf{x}=\mathbf{0}$ contains the point (4,1), the vector $\mathbf{x}=(4,1)$ satisfies $A\mathbf{x}=\mathbf{0}$. Write this equation as a vector equation, using $\mathbf{a}_1$ and $\mathbf{a}_2$ for the columns of A:

$$4\cdot\mathbf{a}_1+1\cdot\mathbf{a}_2=\mathbf{0}$$

Then $\mathbf{a}_2=-4\mathbf{a}_1$. So choose any nonzero vector for the first column of A and multiply that column by – 4 to get the second column of A. For example, set $A=\begin{bmatrix}1&-4\\1&-4\end{bmatrix}$.

Finally, the only way the solution set of $A\mathbf{x}=\mathbf{b}$ could *not* be parallel to the line through (1,4) and the origin is for the solution set of $A\mathbf{x}=\mathbf{b}$ to be *empty*. This does not contradict Theorem 6, because that theorem applies only to the case when the equation $A\mathbf{x}=\mathbf{b}$ has a nonempty solution set. For **b**, take any vector that is *not* a multiple of the columns of A.

Note: In the *Study Guide*, a "Checkpoint" for Section 1.5 will help students with Exercise 37.

38. No. If $A\mathbf{x}=\mathbf{y}$ has no solution, then A cannot have a pivot in each row. Since A is 3×3, it has at most two pivot positions. So the equation $A\mathbf{x}=\mathbf{z}$ for any **z** has at most two basic variables and at least one free variable. Thus, the solution set for $A\mathbf{x}=\mathbf{z}$ is either empty or has infinitely many elements.

39. If **u** satisfies $A\mathbf{x}=\mathbf{0}$, then $A\mathbf{u}=\mathbf{0}$. For any scalar c, Theorem 5(b) in Section 1.4 shows that $A(c\mathbf{u})=cA\mathbf{u}=c\cdot\mathbf{0}=\mathbf{0}$.

40. Suppose $A\mathbf{u}=\mathbf{0}$ and $A\mathbf{v}=\mathbf{0}$. Then, since $A(\mathbf{u}+\mathbf{v})=A\mathbf{u}+A\mathbf{v}$ by Theorem 5(a) in Section 1.4,

$$A(\mathbf{u}+\mathbf{v})=A\mathbf{u}+A\mathbf{v}=\mathbf{0}+\mathbf{0}=\mathbf{0}.$$

Now, let c and d be scalars. Using both parts of Theorem 5,

$$A(c\mathbf{u}+d\mathbf{v})=A(c\mathbf{u})+A(d\mathbf{v})=cA\mathbf{u}+dA\mathbf{v}=c\mathbf{0}+d\mathbf{0}=\mathbf{0}.$$

Note: The MATLAB box in the *Study Guide* introduces the `zeros` command, in order to augment a matrix with a column of zeros.