1.7 SOLUTIONS

Note: Key exercises are 9–20 and 23–30. Exercise 30 states a result that could be a theorem in the text. There is a danger, however, that students will memorize the result without understanding the proof, and then later mix up the words row and column. Exercises 37 and 38 anticipate the discussion in Section 1.9 of one-to-one transformations. Exercise 44 is fairly difficult for my students.

1. Use an augmented matrix to study the solution set of $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$ (*), where **u**, **v**, and **w** are the three given vectors. Since $\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{5} & 7 & 9 & 0 \\ 0 & \textcircled{2} & 4 & 0 \\ 0 & 0 & \textcircled{4} & 0 \end{bmatrix}$, there are no free variables. So the homogeneous equation (*) has only the trivial solution. The vectors are linearly independent.

2. Use an augmented matrix to study the solution set of $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$ (*), where **u**, **v**, and **w** are the three given vectors. Since $\begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 5 & 4 & 0 \\ 2 & -8 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -8 & 1 & 0 \\ 0 & \textcircled{5} & 4 & 0 \\ 0 & 0 & \textcircled{-3} & 0 \end{bmatrix}$, there are no free variables. So the homogeneous equation (*) has only the trivial solution. The vectors are linearly independent.

3. Use the method of Example 3 (or the box following the example). By comparing entries of the vectors, one sees that the second vector is –3 times the first vector. Thus, the two vectors are linearly dependent.

4. From the first entries in the vectors, it seems that the second vector of the pair $\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -8 \end{bmatrix}$ may be 2 times the first vector. But there is a sign problem with the second entries. So neither of the vectors is a multiple of the other. The vectors are linearly independent.

5. Use the method of Example 2. Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 0 & -8 & 5 & 0 \\ 3 & -7 & 4 & 0 \\ -1 & 5 & -4 & 0 \\ 1 & -3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 3 & -7 & 4 & 0 \\ -1 & 5 & -4 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 2 & 0 \\ 0 & \textcircled{2} & -2 & 0 \\ 0 & 0 & \textcircled{-3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are no free variables. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and so the columns of A are linearly independent.

6. Use the method of Example 2. Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ -4 & -3 & 0 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & -3 & 12 & 0 \\ 0 & 4 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 3 & 0 \\ 0 & \textcircled{-1} & 4 & 0 \\ 0 & 0 & \textcircled{7} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are no free variables. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and so the columns of A are linearly independent.

7. Study the equation $A\mathbf{x} = \mathbf{0}$. Some people may start with the method of Example 2:

$$\begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ -2 & -7 & 5 & 1 & 0 \\ -4 & -5 & 7 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 11 & -5 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 4 & -3 & 0 & 0 \\ 0 & \textcircled{1} & -1 & 1 & 0 \\ 0 & 0 & \textcircled{6} & -6 & 0 \end{bmatrix}$$

But this is a waste of time. There are only 3 rows, so there are at most three pivot positions. Hence, at least one of the four variables must be free. So the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and the columns of A are linearly dependent.

8. Same situation as with Exercise 7. The (unnecessary) row operations are

$$\begin{bmatrix} 1 & -3 & 3 & -2 & 0 \\ -3 & 7 & -1 & 2 & 0 \\ 0 & 1 & -4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & -2 & 0 \\ 0 & -2 & 8 & -4 & 0 \\ 0 & 1 & -4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 3 & -2 & 0 \\ 0 & \textcircled{-2} & 8 & -4 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \end{bmatrix}$$

Again, because there are at most three pivot positions yet there are four variables, the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and the columns of A are linearly dependent.

9. **a.** The vector $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ has a solution. To find out, row reduce $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3]$, considered as an augmented matrix:

$$\begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 5 \\ 0 & 0 & \textcircled{8} \\ 0 & 0 & h-10 \end{bmatrix}$$

At this point, the equation $0 = 8$ shows that the original vector equation has no solution. So $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ for *no* value of h.

b. For $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent, the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ must have only the trivial solution. Row reduce the augmented matrix $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{0}]$

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & h-10 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 5 & 0 \\ 0 & 0 & \textcircled{8} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For every value of h, x_2 is a free variable, and so the homogeneous equation has a nontrivial solution. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set for all h.

10. a. The vector $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ has a solution. To find out, row reduce $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$, considered as an augmented matrix:

$$\begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 2 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & h+6 \end{bmatrix}$$

At this point, the equation $0 = 1$ shows that the original vector equation has no solution. So $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ for *no* value of h.

b. For $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent, the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ must have only the trivial solution. Row reduce the augmented matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{0}]$

$$\begin{bmatrix} 1 & -2 & 2 & 0 \\ -5 & 10 & -9 & 0 \\ -3 & 6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h+6 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 2 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For every value of h, x_2 is a free variable, and so the homogeneous equation has a nontrivial solution. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set for all h.

11. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{0}]$:

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ -1 & -5 & 5 & 0 \\ 4 & 7 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -5 & h+4 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & -1 & 0 \\ 0 & \textcircled{-2} & 4 & 0 \\ 0 & 0 & h-6 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution if and only if $h - 6 = 0$ (which corresponds to x_3 being a free variable). Thus, the vectors are linearly dependent if and only if $h = 6$.

12. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{0}]$:

$$\begin{bmatrix} 2 & -6 & 8 & 0 \\ -4 & 7 & h & 0 \\ 1 & -3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -6 & 8 & 0 \\ 0 & \textcircled{-5} & h+16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a free variable and hence a nontrivial solution no matter what the value of h. So the vectors are linearly dependent for all values of h.

13. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{0}]$:

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -9 & h & 0 \\ -3 & 6 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 3 & 0 \\ 0 & \textcircled{1} & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a free variable and hence a nontrivial solution no matter what the value of h. So the vectors are linearly dependent for all values of h.

14. To study the linear dependence of three vectors, say $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3\ \ \mathbf{0}]$:

$$\begin{bmatrix} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ -3 & 8 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -7 & h+3 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -5 & 1 & 0 \\ 0 & \textcircled{2} & 2 & 0 \\ 0 & 0 & h+10 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution if and only if $h + 10 = 0$ (which corresponds to x_3 being a free variable). Thus, the vectors are linearly dependent if and only if $h = -10$.

15. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.

16. The set is linearly dependent because the second vector is 3/2 times the first vector.

17. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.

18. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.

19. The set is linearly independent because neither vector is a multiple of the other vector. [Two of the entries in the first vector are – 4 times the corresponding entry in the second vector. But this multiple does not work for the third entries.]

20. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.

21. **a.** False. A homogeneous system *always* has the trivial solution. See the box before Example 2.

b. False. See the warning after Theorem 7.

c. True. See Fig. 3, after Theorem 8.

d. True. See the remark following Example 4.

22. **a.** True. See Fig. 1.

b. False. For instance, the set consisting of $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$ is linearly dependent. See the warning after Theorem 8.

c. True. See the remark following Example 4.

d. False. See Example 3(a).

23. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$

24. $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

25. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

26. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$. The columns must linearly independent, by Theorem 7, because the first column is not zero, the second column is not a multiple of the first, and the third column is not a linear combination of the preceding two columns (because $\mathbf{a}_3$ is not in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$).

27. All five columns of the 7×5 matrix A must be pivot columns. Otherwise, the equation $A\mathbf{x} = \mathbf{0}$ would have a free variable, in which case the columns of A would be linearly dependent.

28. If the columns of a 5×7 matrix A span $\mathbf{R}^5$, then A has a pivot in each row, by Theorem 4. Since each pivot position is in a different column, A has five pivot columns.

29. A: any 3×2 matrix with two nonzero columns such that neither column is a multiple of the other. In this case the columns are linearly independent and so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
B: any 3×2 matrix with one column a multiple of the other.

30. **a.** n

b. The columns of A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens if and only if $A\mathbf{x} = \mathbf{0}$ has no free variables, which in turn happens if and only if every variable is a basic variable, that is, if and only if every column of A is a pivot column.

31. Think of $A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3]$. The text points out that $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$. Rewrite this as $\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. As a matrix equation, $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = (1, 1, -1)$.

32. Think of $A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3]$. The text points out that $\mathbf{a}_1 + 2\mathbf{a}_2 = \mathbf{a}_3$. Rewrite this as $\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. As a matrix equation, $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = (1, 2, -1)$.

33. True, by Theorem 7. (The *Study Guide* adds another justification.)

34. True, by Theorem 9.

35. False. The vector $\mathbf{v}_1$ could be the zero vector.

36. False. Counterexample: Take $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$ all to be multiples of one vector. Take $\mathbf{v}_3$ to be *not* a multiple of that vector. For example,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

37. True. A linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ may be extended to a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ by placing a zero weight on $\mathbf{v}_4$.

38. True. If the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ had a nontrivial solution (with at least one of x_1, x_2, x_3 nonzero), then so would the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0\cdot\mathbf{v}_4 = \mathbf{0}$. But that cannot happen because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be linearly independent. This problem can also be solved using Exercise 37, if you know that the statement there is true.

39. If for all **b** the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution, then take $\mathbf{b} = \mathbf{0}$, and conclude that the equation $A\mathbf{x} = \mathbf{0}$ has at most one solution. Then the trivial solution is the only solution, and so the columns of A are linearly independent.

40. An $m \times n$ matrix with n pivot columns has a pivot in each column. So the equation $A\mathbf{x} = \mathbf{b}$ has no free variables. If there is a solution, it must be unique.

41. [M] $A = \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ 0 & 5/8 & 5 & 25/8 & -19/4 \\ 0 & 1/4 & 2 & 5/4 & 5/2 \\ 0 & 7/8 & 7 & 35/8 & 35/4 \end{bmatrix}$

$\sim \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ 0 & 5/8 & 5 & 25/8 & -19/4 \\ 0 & 0 & 0 & 0 & 22/5 \\ 0 & 0 & 0 & 0 & 77/5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{8} & -3 & 0 & -7 & 2 \\ 0 & \textcircled{5/8} & 5 & 25/8 & -19/4 \\ 0 & 0 & 0 & 0 & \textcircled{22/5} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

The pivot columns of A are 1, 2, and 5. Use them to form $B = \begin{bmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix}$.

Other likely choices use columns 3 or 4 of A instead of 2: $\begin{bmatrix} 8 & 0 & 2 \\ -9 & 5 & -7 \\ 6 & 2 & 4 \\ 5 & 7 & 10 \end{bmatrix}, \begin{bmatrix} 8 & -7 & 2 \\ -9 & 11 & -7 \\ 6 & -4 & 4 \\ 5 & 0 & 10 \end{bmatrix}$.

Actually, any set of three columns of A that includes column 5 will work for B, but the concepts needed to prove that are not available now. (Column 5 is not in the two-dimensional subspace spanned by the first four columns.)

42. [M]

$\begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{12} & 10 & -6 & -3 & 7 & 10 \\ 0 & \textcircled{-1/6} & 1/2 & 21/4 & -59/12 & 65/6 \\ 0 & 0 & 0 & \textcircled{89/2} & -89/2 & 89 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

The pivot columns of A are 1, 2, 4, and 6. Use them to form $B = \begin{bmatrix} 12 & 10 & -3 & 10 \\ -7 & -6 & 7 & 5 \\ 9 & 9 & -5 & -1 \\ -4 & -3 & 6 & 9 \\ 8 & 7 & -9 & -8 \end{bmatrix}$.

Other likely choices might use column 3 of A instead of 2, and/or use column 5 instead of 4.

43. **[M]** Make **v** any one of the columns of A that is not in B and row reduce the augmented matrix $[B \quad \mathbf{v}]$. The calculations will show that the equation $B\mathbf{x} = \mathbf{v}$ is consistent, which means that **v** is a linear combination of the columns of B. Thus, each column of A that is not a column of B is in the set spanned by the columns of B.

44. **[M]** Calculations made as for Exercise 43 will show that each column of A that is not a column of B is in the set spanned by the columns of B. *Reason*: The original matrix A has only four pivot columns. If one or more columns of A are removed, the resulting matrix will have at most four pivot columns. (Use exactly the same row operations on the new matrix that were used to reduce A to echelon form.) If **v** is a column of A that is not in B, then row reduction of the augmented matrix $[B \quad \mathbf{v}]$ will display at most four pivot columns. Since B itself was constructed to have four pivot columns, adjoining **v** cannot produce a fifth pivot column. Thus the first four columns of $[B \quad \mathbf{v}]$ are the pivot columns. This implies that the equation $B\mathbf{x} = \mathbf{v}$ has a solution.

Note: At the end of Section 1.7, the *Study Guide* has another note to students about "Mastering Linear Algebra Concepts." The note describes how to organize a review sheet that will help students form a mental image of linear independence. The note also lists typical misuses of terminology, in which an adjective is applied to an inappropriate noun. (This is a major problem for my students.) I require my students to prepare a review sheet as described in the *Study Guide*, and I try to make helpful comments on their sheets. I am convinced, through personal observation and student surveys, that the students who prepare many of these review sheets consistently perform better than other students. Hopefully, these students will remember important concepts for some time beyond the final exam.