1.8 SOLUTIONS

Notes: The key exercises are 17–20, 25 and 31. Exercise 20 is worth assigning even if you normally assign only odd exercises. Exercise 25 (and 27) can be used to make a few comments about computer graphics, even if you do not plan to cover Section 2.6. For Exercise 31, the *Study Guide* encourages students *not* to look at the proof before trying hard to construct it. Then the *Guide* explains how to create the proof.

Exercises 19 and 20 provide a natural segue into Section 1.9. I arrange to discuss the homework on these exercises when I am ready to begin Section 1.9. The definition of the standard matrix in Section 1.9 follows naturally from the homework, and so I've covered the first page of Section 1.9 before students realize we are working on new material.

The text does not provide much practice determining whether a transformation is linear, because the time needed to develop this skill would have to be taken away from some other topic. If you want your students to be able to do this, you may need to supplement Exercises 29, 30, 32 and 33.

If you skip the concepts of one-to-one and "onto" in Section 1.9, you can use the result of Exercise 31 to show that the coordinate mapping from a vector space onto $\mathbf{R}^n$ (in Section 4.4) preserves linear independence and dependence of sets of vectors. (See Example 6 in Section 4.4.)

1. $T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$

2. $T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} .5 \\ 0 \\ -2 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} .5a \\ .5b \\ .5c \end{bmatrix}$

3. $[A \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, unique solution

4. $[A \quad \mathbf{b}] = \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$, unique solution

5. $[A \quad \mathbf{b}] = \begin{bmatrix} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 3 & 3 \\ 0 & \textcircled{1} & 2 & 1 \end{bmatrix}$

Note that a solution is *not* $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. To avoid this common error, write the equations:

$$\begin{array}{rcrcr} \textcircled{x_1} & & + \; 3x_3 & = & 3 \\ & \textcircled{x_2} & + \; 2x_3 & = & 1 \end{array}$$ and solve for the basic variables: $\begin{cases} x_1 = 3 - 3x_3 \\ x_2 = 1 - 2x_3 \\ x_3 \text{ is free} \end{cases}$

General solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 3x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$. For a particular solution, one might choose

$x_3 = 0$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

6. $[A \quad \mathbf{b}] = \begin{bmatrix} 1 & -2 & 1 & 1 \\ 3 & -4 & 5 & 9 \\ 0 & 1 & 1 & 3 \\ -3 & 5 & -4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 2 & 2 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 3 & 7 \\ 0 & \textcircled{1} & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{array}{rcrcr} \textcircled{x_1} & & + \; 3x_3 & = & 7 \\ & \textcircled{x_2} & + \; x_3 & = & 3 \end{array}. \quad \begin{cases} x_1 = 7 - 3x_3 \\ x_2 = 3 - x_3 \\ x_3 \text{ is free} \end{cases}$$

General solution: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 - 3x_3 \\ 3 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$, one choice: $\begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$.

7. $a = 5$; the domain of T is $\mathbf{R}^5$, because a 6×5 matrix has 5 columns and for $A\mathbf{x}$ to be defined, $\mathbf{x}$ must be in $\mathbf{R}^5$. $b = 6$; the codomain of T is $\mathbf{R}^6$, because $A\mathbf{x}$ is a linear combination of the columns of A, and each column of A is in $\mathbf{R}^6$.

8. A must have 5 rows and 4 columns. For the domain of T to be $\mathbf{R}^4$, A must have four columns so that $A\mathbf{x}$ is defined for $\mathbf{x}$ in $\mathbf{R}^4$. For the codomain of T to be $\mathbf{R}^5$, the columns of A must have five entries (in which case A must have five rows), because $A\mathbf{x}$ is a linear combination of the columns of A.

9. Solve $A\mathbf{x} = \mathbf{0}$. $\begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 2 & -8 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} \textcircled{1} & 0 & -9 & 7 & 0 \\ 0 & \textcircled{1} & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \textcircled{x_1} & & - 9x_3 + 7x_4 = 0 \\ & \textcircled{x_2} & - 4x_3 + 3x_4 = 0, \\ & & 0 = 0 \end{matrix} \quad \begin{cases} x_1 = 9x_3 - 7x_4 \\ x_2 = 4x_3 - 3x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9x_3 - 7x_4 \\ 4x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

10. Solve $A\mathbf{x} = \mathbf{0}$. $\begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 3 & 0 & 0 \\ 0 & \textcircled{1} & 2 & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} \textcircled{x_1} + 3x_3 = 0 \\ \textcircled{x_2} + 2x_3 = 0 \\ \textcircled{x_4} = 0 \end{matrix} \quad \begin{cases} x_1 = -3x_3 \\ x_2 = -2x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases} \quad \mathbf{x} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

11. Is the system represented by $[A \;\; \mathbf{b}]$ consistent? Yes, as the following calculation shows.

$$\begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 2 & -8 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -4 & 7 & -5 & -1 \\ 0 & \textcircled{1} & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent, so $\mathbf{b}$ is in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

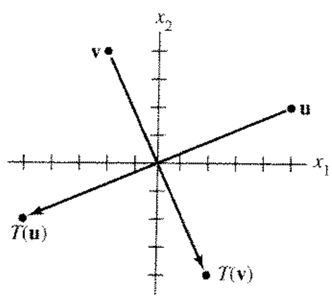
12. Is the system represented by $[A \;\; \mathbf{b}]$ consistent?

$$\begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 9 & 18 & 9 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 9 & 18 & 9 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -18 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 17 \end{bmatrix}$$

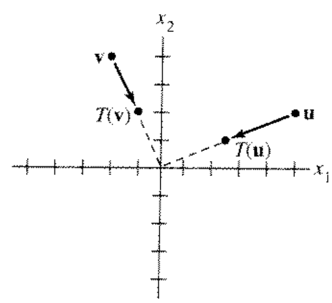
The system is inconsistent, so $\mathbf{b}$ is not in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

13.



A reflection through the origin.

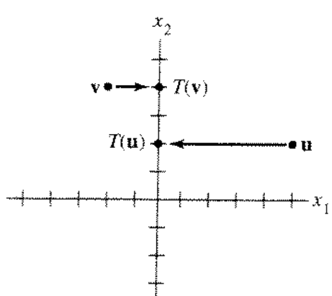
14.



A contraction by the factor .5.

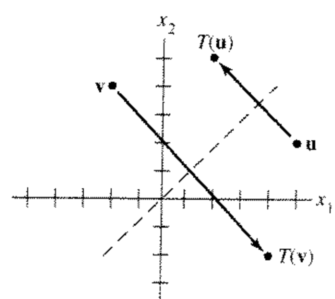
The transformation in Exercise 13 may also be described as a rotation of π radians about the origin or a rotation of $-\pi$ radians about the origin.

15.



A projection onto the x_2-axis

16.

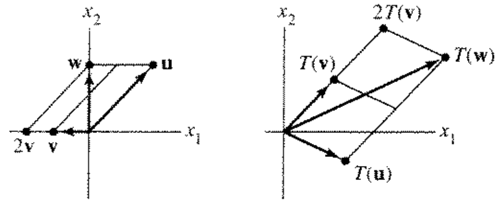


A reflection through the line $x_2 = x_1$.

17. $T(3\mathbf{u}) = 3T(\mathbf{u}) = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$, $T(2\mathbf{v}) = 2T(\mathbf{v}) = 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$, and

$$T(3\mathbf{u} + 2\mathbf{v}) = 3T(\mathbf{u}) = 2T(\mathbf{v}) = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}.$$

18. Draw a line through **w** parallel to **v**, and draw a line through **w** parallel to **u**. See the left part of the figure below. From this, estimate that $\mathbf{w} = \mathbf{u} + 2\mathbf{v}$. Since T is linear, $T(\mathbf{w}) = T(\mathbf{u}) + 2T(\mathbf{v})$. Locate $T(\mathbf{u})$ and $2T(\mathbf{v})$ as in the right part of the figure and form the associated parallelogram to locate $T(\mathbf{w})$.



19. All we know are the images of $\mathbf{e}_1$ and $\mathbf{e}_2$ and the fact that T is linear. The key idea is to write $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5\mathbf{e}_1 - 3\mathbf{e}_2$. Then, from the linearity of T, write

$$T(\mathbf{x}) = T(5\mathbf{e}_1 - 3\mathbf{e}_2) = 5T(\mathbf{e}_1) - 3T(\mathbf{e}_2) = 5\mathbf{y}_1 - 3\mathbf{y}_2 = 5\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}.$$

To find the image of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, observe that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. Then

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) = x_1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

20. Use the basic definition of $A\mathbf{x}$ to construct A. Write

$$T(\mathbf{x}) = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}\mathbf{x}, \quad A = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$$

21. **a.** True. Functions from $\mathbf{R}^n$ to $\mathbf{R}^m$ are defined before Fig. 2. A linear transformation is a function with certain properties.

b. False. The domain is $\mathbf{R}^5$. See the paragraph before Example 1.

c. False. The range is the set of all linear combinations of the columns of A. See the paragraph before Example 1.

d. False. See the paragraph after the definition of a linear transformation.

e. True. See the paragraph following the box that contains equation (4).

22. **a.** True. See the paragraph following the definition of a linear transformation.

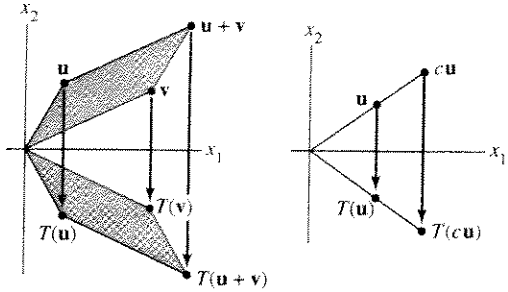
b. False. If A is an $m \times n$ matrix, the codomain is $\mathbf{R}^m$. See the paragraph before Example 1.

c. False. The question is an existence question. See the remark about Example 1(d), following the solution of Example 1.

d. True. See the discussion following the definition of a linear transformation.

e. True. See the paragraph following equation (5).

23.



24. Given any $\mathbf{x}$ in $\mathbf{R}^n$, there are constants $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_p\mathbf{v}_p$, because $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbf{R}^n$. Then, from property (5) of a linear transformation,

$$T(\mathbf{x}) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) = c_1\mathbf{0} + \cdots + c_p\mathbf{0} = \mathbf{0}$$

25. Any point $\mathbf{x}$ on the line through $\mathbf{p}$ in the direction of $\mathbf{v}$ satisfies the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ for some value of t. By linearity, the image $T(\mathbf{x})$ satisfies the parametric equation

$$T(\mathbf{x}) = T(\mathbf{p} + t\mathbf{v}) = T(\mathbf{p}) + tT(\mathbf{v}) \qquad (*)$$

If $T(\mathbf{v}) = \mathbf{0}$, then $T(\mathbf{x}) = T(\mathbf{p})$ for all values of t, and the image of the original line is just a single point. Otherwise, (*) is the parametric equation of a line through $T(\mathbf{p})$ in the direction of $T(\mathbf{v})$.

26. Any point $\mathbf{x}$ on the plane P satisfies the parametric equation $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ for some values of s and t. By linearity, the image $T(\mathbf{x})$ satisfies the parametric equation

$$T(\mathbf{x}) = sT(\mathbf{u}) + tT(\mathbf{v}) \qquad (s, t \text{ in } \mathbf{R}) \qquad (*)$$

The set of images is just Span$\{T(\mathbf{u}), T(\mathbf{v})\}$. If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly independent, Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is a plane through $T(\mathbf{u})$, $T(\mathbf{v})$, and $\mathbf{0}$. If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent and not both zero, then Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is a line through $\mathbf{0}$. If $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{0}$, then Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is $\{\mathbf{0}\}$.

27. **a.** From Fig. 7 in the exercises for Section 1.5, the line through $T(\mathbf{p})$ and $T(\mathbf{q})$ is in the direction of $\mathbf{q} - \mathbf{p}$, and so the equation of the line is $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \mathbf{p} + t\mathbf{q} - t\mathbf{p} = (1 - t)\mathbf{p} + t\mathbf{q}$.

b. Consider $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$ for t such that $0 \le t \le 1$. Then, by linearity of T,

$$T(\mathbf{x}) = T((1 - t)\mathbf{p} + t\mathbf{q}) = (1 - t)T(\mathbf{p}) + tT(\mathbf{q}) \qquad 0 \le t \le 1 \qquad (*)$$

If $T(\mathbf{p})$ and $T(\mathbf{q})$ are distinct, then (*) is the equation for the line segment between $T(\mathbf{p})$ and $T(\mathbf{q})$, as shown in part (a) Otherwise, the set of images is just the single point $T(\mathbf{p})$, because

$$(1 - t)T(\mathbf{p}) + tT(\mathbf{q}) = (1 - t)T(\mathbf{p}) + tT(\mathbf{p}) = T(\mathbf{p})$$

28. Consider a point $\mathbf{x}$ in the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, say $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$ for $0 \le a \le 1$, $0 \le b \le 1$. By linearity of T, the image of $\mathbf{x}$ is

$$T(\mathbf{x}) = T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v}), \text{ for } 0 \le a \le 1, 0 \le b \le 1 \qquad (*)$$

This image point lies in the parallelogram determined by $T(\mathbf{u})$ and $T(\mathbf{v})$.

Special "degenerate" cases arise when $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent. If one of the images is not zero, then the "parallelogram" is actually the line segment from $\mathbf{0}$ to $T(\mathbf{u}) + T(\mathbf{v})$. If both $T(\mathbf{u})$ and $T(\mathbf{v})$ are zero, then the parallelogram is just $\{\mathbf{0}\}$. Another possibility is that even $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, in which case the original parallelogram is degenerate (either a line segment or the zero vector). In this case, the set of images must be degenerate, too.

29. **a.** When $b = 0, f(x) = mx$. In this case, for all x,y in $\mathbf{R}$ and all scalars c and d,

$$f(cx + dy) = m(cx + dy) = mcx + mdy = c(mx) + d(my) = c \cdot f(x) + d \cdot f(y)$$

This shows that f is linear.

b. When $f(x) = mx + b$, with b nonzero, $f(0) = m(0) = b = b \neq 0$. This shows that f is not linear, because every linear transformation maps the zero vector in its domain into the zero vector in the codomain. (In this case, both zero vectors are just the number 0.) Another argument, for instance, would be to calculate $f(2x) = m(2x) + b$ and $2f(x) = 2mx + 2b$. If b is nonzero, then $f(2x)$ is not equal to $2f(x)$ and so f is not a linear transformation.

c. In calculus, f is called a "linear function" because the graph of f is a line.

30. Let $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for $\mathbf{x}$ in $\mathbf{R}^n$. If $\mathbf{b}$ is not zero, $T(\mathbf{0}) = A\mathbf{0} + \mathbf{b} = \mathbf{b} \neq \mathbf{0}$. Actually, T fails both properties of a linear transformation. For instance, $T(2\mathbf{x}) = A(2\mathbf{x}) + \mathbf{b} = 2A\mathbf{x} + \mathbf{b}$, which is not the same as $2T(\mathbf{x}) = 2(A\mathbf{x} + \mathbf{b}) = 2A\mathbf{x} + 2\mathbf{b}$. Also,

$$T(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) + \mathbf{b} = A\mathbf{x} + A\mathbf{y} + \mathbf{b}$$

which is not the same as

$$T(\mathbf{x}) + T(\mathbf{y}) = A\mathbf{x} + \mathbf{b} + A\mathbf{y} + \mathbf{b}$$

31. (The *Study Guide* has a more detailed discussion of the proof.) Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Then there exist scalars c_1, c_2, c_3, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

Then $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = T(\mathbf{0}) = \mathbf{0}$. Since T is linear,

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \mathbf{0}$$

Since not all the weights are zero, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is a linearly dependent set.

32. Take any vector (x_1, x_2) with $x_2 \neq 0$, and use a negative scalar. For instance, $T(0, 1) = (-2, 3)$, but $T(-1\cdot(0, 1)) = T(0, -1) = (2, 3) \neq (-1)\cdot T(0, 1)$.

33. One possibility is to show that T does not map the zero vector into the zero vector, something that every linear transformation *does* do. $T(0, 0) = (0, 4, 0)$.

34. Suppose that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set in $\mathbf{R}^n$ and yet $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent. Then there exist weights c_1, c_2, not both zero, such that

$$c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$$

Because T is linear, $T(c_1\mathbf{u} + c_2\mathbf{v}) = \mathbf{0}$. That is, the vector $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$ satisfies $T(\mathbf{x}) = \mathbf{0}$. Furthermore, $\mathbf{x}$ cannot be the zero vector, since that would mean that a nontrivial linear combination of $\mathbf{u}$ and $\mathbf{v}$ is zero, which is impossible because $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Thus, the equation $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution.

35. Take $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^3$ and let c and d be scalars. Then

$c\mathbf{u} + d\mathbf{v} = (cu_1 + dv_1, cu_2 + dv_2, cu_3 + dv_3)$. The transformation T is linear because

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= (cu_1 + dv_1, cu_2 + dv_2, -(cu_3 + dv_3)) = (cu_1 + dv_1, cu_2 + dv_2, cu_3\;\; dv_3) \\ &= (cu_1, cu_2, \;\; cu_3) + (dv_1, dv_2, \;\; dv_3) = c(u_1, u_2, \;\; u_3) + d(v_1, v_2, \;\; v_3) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

36. Take $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^3$ and let c and d be scalars. Then

$c\mathbf{u} + d\mathbf{v} = (cu_1 + dv_1, cu_2 + dv_2, cu_3 + dv_3)$. The transformation T is linear because

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= (cu_1 + dv_1, 0, cu_3 + dv_3) = (cu_1, 0, cu_3) + (dv_1, 0, dv_3) \\ &= c(u_1, 0, u_3) + d(v_1, 0, v_3) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

37. [M] $\begin{bmatrix} 4 & -2 & 5 & -5 & 0 \\ -9 & 7 & -8 & 0 & 0 \\ -6 & 4 & 5 & 3 & 0 \\ 5 & -3 & 8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & -7/2 & 0 \\ 0 & \textcircled{1} & 0 & -9/2 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{cases} x_1 = (7/2)x_4 \\ x_2 = (9/2)x_4 \\ x_3 = 0 \\ x_4 \text{ is free} \end{cases}$ $\mathbf{x} = x_4 \begin{bmatrix} 7/2 \\ 9/2 \\ 0 \\ 1 \end{bmatrix}$

38. [M] $\begin{bmatrix} -9 & -4 & -9 & 4 & 0 \\ 5 & -8 & -7 & 6 & 0 \\ 7 & 11 & 16 & -9 & 0 \\ 9 & -7 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 3/4 & 0 \\ 0 & \textcircled{1} & 0 & 5/4 & 0 \\ 0 & 0 & \textcircled{1} & -7/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{cases} x_1 = -(3/4)x_4 \\ x_2 = -(5/4)x_4 \\ x_3 = (7/4)x_4 \\ x_4 \text{ is free} \end{cases}$ $\mathbf{x} = x_4 \begin{bmatrix} -3/4 \\ -5/4 \\ 7/4 \\ 1 \end{bmatrix}$

39. [M] $\begin{bmatrix} 4 & -2 & 5 & -5 & 7 \\ -9 & 7 & -8 & 0 & 5 \\ -6 & 4 & 5 & 3 & 9 \\ 5 & -3 & 8 & -4 & 7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & -7/2 & 4 \\ 0 & \textcircled{1} & 0 & -9/2 & 7 \\ 0 & 0 & \textcircled{1} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, yes, **b** is in the range of the transformation, because the augmented matrix shows a consistent system. In fact,

the general solution is $\begin{cases} x_1 = 4 + (7/2)x_4 \\ x_2 = 7 + (9/2)x_4 \\ x_3 = 1 \\ x_4 \text{ is free} \end{cases}$; when $x_4 = 0$ a solution is $\mathbf{x} = \begin{bmatrix} 4 \\ 7 \\ 1 \\ 0 \end{bmatrix}$.

40. [M] $\begin{bmatrix} -9 & -4 & -9 & 4 & -7 \\ 5 & -8 & -7 & 6 & -7 \\ 7 & 11 & 16 & -9 & 13 \\ 9 & -7 & -4 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 3/4 & -5/4 \\ 0 & \textcircled{1} & 0 & 5/4 & -11/4 \\ 0 & 0 & \textcircled{1} & -7/4 & 13/4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, yes, **b** is in the range of the transformation, because the augmented matrix shows a consistent system. In fact,

the general solution is $\begin{cases} x_1 = -5/4 - (3/4)x_4 \\ x_2 = -11/4 - (5/4)x_4 \\ x_3 = 13/4 + (7/4)x_4 \\ x_4 \text{ is free} \end{cases}$; when $x_4 = 1$ a solution is $\mathbf{x} = \begin{bmatrix} -2 \\ -4 \\ 5 \\ 1 \end{bmatrix}$.

Notes: At the end of Section 1.8, the *Study Guide* provides a list of equations, figures, examples, and connections with concepts that will strengthen a student's understanding of linear transformations. I encourage my students to continue the construction of review sheets similar to those for "span" and "linear independence," but I refrain from collecting these sheets. At some point the students have to assume the responsibility for mastering this material.

If your students are using MATLAB or another matrix program, you might insert the definition of matrix multiplication after this section, and then assign a project that uses random matrices to explore properties of matrix multiplication. See Exercises 34–36 in Section 2.1. Meanwhile, in class you can continue with your plans for finishing Chapter 1. When you get to Section 2.1, you won't have much to do. The *Study Guide*'s MATLAB note for Section 2.1 contains the matrix notation students will need for a project on matrix multiplication. The appendices in the *Study Guide* have the corresponding material for Mathematica, Maple, and the T-83+/86/89 and HP-48G graphic calculators.