Chapter 1 SUPPLEMENTARY EXERCISES

1. **a.** False. (The word "reduced" is missing.) Counterexample:

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix},\ B=\begin{bmatrix}1 & 2\\ 0 & -2\end{bmatrix},\ C=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

The matrix A is row equivalent to matrices B and C, both in echelon form.

b. False. Counterexample: Let A be any $n \times n$ matrix with fewer than n pivot columns. Then the equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. (Theorem 2 in Section 1.2 says that a system has either zero, one, or infinitely many solutions, but it does not say that a system with infinitely many solutions exists. Some counterexample is needed.)

c. True. If a linear system has more than one solution, it is a consistent system and has a free variable. By the Existence and Uniqueness Theorem in Section 1.2, the system has infinitely many solutions.

d. False. Counterexample: The following system has no free variables and no solution:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_2 &= 5 \\ x_1 + x_2 &= 2 \end{aligned}$$

e. True. See the box after the definition of elementary row operations, in Section 1.1. If $[A \ \ \mathbf{b}]$ is transformed into $[C \ \ \mathbf{d}]$ by elementary row operations, then the two augmented matrices are row equivalent.

f. True. Theorem 6 in Section 1.5 essentially says that when $A\mathbf{x} = \mathbf{b}$ is consistent, the solution sets of the nonhomogeneous equation and the homogeneous equation are translates of each other. In this case, the two equations have the same number of solutions.

g. False. For the columns of A to span $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ must be consistent for *all* $\mathbf{b}$ in $\mathbf{R}^m$, not for just one vector $\mathbf{b}$ in $\mathbf{R}^m$.

h. False. *Any* matrix can be transformed by elementary row operations into reduced echelon form, but not every matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent.

i. True. If A is row equivalent to B, then A can be transformed by elementary row operations first into B and then further transformed into the reduced echelon form U of B. Since the reduced echelon form of A is unique, it must be U.

j. False. Every equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution whether or not some variables are free.

k. True, by Theorem 4 in Section 1.4. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbf{R}^m$, then A must have a position in every one of its m rows. If A has m pivot positions, then A has m pivot columns, each containing one pivot position.

l. False. The word "unique" should be deleted. Let A be any matrix with m pivot columns but more than m columns altogether. Then the equation $A\mathbf{x} = \mathbf{b}$ is consistent and has m basic variables and at least one free variable. Thus the equation does not does not have a unique solution.

m. True. If A has n pivot positions, it has a pivot in each of its n columns and in each of its n rows. The reduced echelon form has a 1 in each pivot position, so the reduced echelon form is the $n \times n$ identity matrix.

n. True. Both matrices A and B can be row reduced to the 3×3 identity matrix, as discussed in the previous question. Since the row operations that transform B into I_3 are reversible, A can be transformed first into I_3 and then into B.

o. True. The reason is essentially the same as that given for question f.

p. True. If the columns of A span $\mathbf{R}^m$, then the reduced echelon form of A is a matrix U with a pivot in each row, by Theorem 4 in Section 1.4. Since B is row equivalent to A, B can be transformed by row operations first into A and then further transformed into U. Since U has a pivot in each row, so does B. By Theorem 4, the columns of B span $\mathbf{R}^m$.

q. False. See Example 5 in Section 1.6.

r. True. Any set of three vectors in $\mathbf{R}^2$ would have to be linearly dependent, by Theorem 8 in Section 1.6.

s. False. If a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ were to span $\mathbf{R}^5$, then the matrix $A = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{v}_4]$ would have a pivot position in each of its five rows, which is impossible since A has only four columns.

t. True. The vector $-\mathbf{u}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, namely, $-\mathbf{u} = (-1)\mathbf{u} + 0\mathbf{v}$.

u. False. If $\mathbf{u}$ and $\mathbf{v}$ are multiples, then Span$\{\mathbf{u}, \mathbf{v}\}$ is a line, and $\mathbf{w}$ need not be on that line.

v. False. Let $\mathbf{u}$ and $\mathbf{v}$ be any linearly independent pair of vectors and let $\mathbf{w} = 2\mathbf{v}$. Then $\mathbf{w} = 0\mathbf{u} + 2\mathbf{v}$, so $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. However, $\mathbf{u}$ cannot be a linear combination of $\mathbf{v}$ and $\mathbf{w}$ because if it were, $\mathbf{u}$ would be a multiple of $\mathbf{v}$. That is not possible since $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.

w. False. The statement would be true if the condition $\mathbf{v}_1$ is not zero were present. See Theorem 7 in Section 1.7. However, if $\mathbf{v}_1 = \mathbf{0}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, no matter what else might be true about $\mathbf{v}_2$ and $\mathbf{v}_3$.

x. True. "Function" is another word used for "transformation" (as mentioned in the definition of "transformation" in Section 1.8), and a linear transformation is a special type of transformation.

y. True. For the transformation $\mathbf{x} \mapsto A\mathbf{x}$ to map $\mathbf{R}^5$ onto $\mathbf{R}^6$, the matrix A would have to have a pivot in every row and hence have six pivot columns. This is impossible because A has only five columns.

z. False. For the transformation $\mathbf{x} \mapsto A\mathbf{x}$ to be one-to-one, A must have a pivot in each column. Since A has n columns and m pivots, m might be less than n.

2. If $a \neq 0$, then $x = b/a$; the solution is unique. If $a = 0$, and $b \neq 0$, the solution set is empty, because $0x = 0 \neq b$. If $a = 0$ and $b = 0$, the equation $0x = 0$ has infinitely many solutions.

3. a. Any consistent linear system whose echelon form is

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b. Any consistent linear system whose coefficient matrix has reduced echelon form I_3.

c. Any inconsistent linear system of three equations in three variables.

4. Since there are three pivots (one in each row), the augmented matrix must reduce to the form $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}$. A solution of $A\mathbf{x} = \mathbf{b}$ exists for all $\mathbf{b}$ because there is a pivot in each row of A. Each solution is unique because there are no free variables.

5. a. $\begin{bmatrix} 1 & 3 & k \\ 4 & h & 8 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & k \\ 0 & h-12 & 8-4k \end{bmatrix}$. If $h = 12$ and $k \neq 2$, the second row of the augmented matrix indicates an inconsistent system of the form $0x_2 = b$, with b nonzero. If $h = 12$, and $k = 2$, there is only one nonzero equation, and the system has infinitely many solutions. Finally, if $h \neq 12$, the coefficient matrix has two pivots and the system has a unique solution.

b. $\begin{bmatrix} -2 & h & 1 \\ 6 & k & -2 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-2} & h & 1 \\ 0 & k+3h & 1 \end{bmatrix}$. If $k + 3h = 0$, the system is inconsistent. Otherwise, the coefficient matrix has two pivots and the system has a unique solution.

6. a. Set $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. "Determine if **b** is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$." Or, "Determine if **b** is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$." To do this, compute

$$\begin{bmatrix} 4 & -2 & 7 & -5 \\ 8 & -3 & 10 & -3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{4} & -2 & 7 & -5 \\ 0 & \textcircled{1} & -4 & 7 \end{bmatrix}$$

The system is consistent, so **b** *is* in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b. Set $A = \begin{bmatrix} 4 & -2 & 7 \\ 8 & -3 & 10 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. "Determine if **b** is a linear combination of the columns of A."

c. Define $T(\mathbf{x}) = A\mathbf{x}$. "Determine if **b** is in the range of T."

7. a. Set $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. "Determine if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span $\mathbf{R}^3$." To do this, row reduce $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3]$:

$$\begin{bmatrix} 2 & -4 & -2 \\ -5 & 1 & 1 \\ 7 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 \\ 0 & -9 & -4 \\ 0 & 9 & 4 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -4 & -2 \\ 0 & \textcircled{-9} & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix does not have a pivot in each row, so its columns do not span $\mathbf{R}^3$, by Theorem 4 in Section 1.4.

b. Set $A = \begin{bmatrix} 2 & -4 & -2 \\ -5 & 1 & 1 \\ 7 & -5 & -3 \end{bmatrix}$. "Determine if the columns of A span $\mathbf{R}^3$."

c. Define $T(\mathbf{x}) = A\mathbf{x}$. "Determine if T maps $\mathbf{R}^3$ onto $\mathbf{R}^3$."

8. a. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$ **b.** $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$

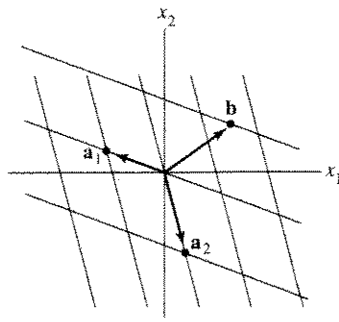
9. The first line is the line spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The second line is spanned by $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So the problem is to write $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ as the sum of a multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and a multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. That is, find x_1 and x_2 such that

$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Reduce the augmented matrix for this equation:

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 0 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 7/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & 7/3 \end{bmatrix}$$

Thus, $\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \frac{4}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{7}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 7/3 \\ 14/3 \end{bmatrix}$.

10. The line through $\mathbf{a}_1$ and the origin and the line through $\mathbf{a}_2$ and the origin determine a "grid" on the x_1x_2-plane as shown below. Every point in $\mathbf{R}^2$ can be described uniquely in terms of this grid. Thus, **b** can be reached from the origin by traveling a certain number of units in the $\mathbf{a}_1$-direction and a certain number of units in the $\mathbf{a}_2$-direction.



11. A solution set is a line when the system has one free variable. If the coefficient matrix is 2×3, then two of the columns should be pivot columns. For instance, take $\begin{bmatrix} 1 & 2 & * \\ 0 & 3 & * \end{bmatrix}$. Put anything in column 3. The resulting matrix will be in echelon form. Make one row replacement operation on the second row to create a matrix *not* in echelon form, such as $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 2 \end{bmatrix}$

12. A solution set is a plane where there are two free variables. If the coefficient matrix is 2×3, then only one column can be a pivot column. The echelon form will have all zeros in the second row. Use a row replacement to create a matrix not in echelon form. For instance, let A $= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$.

13. The reduced echelon form of A looks like $E = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$. Since E is row equivalent to A, the equation $E\mathbf{x} = \mathbf{0}$ has the same solutions as $A\mathbf{x} = \mathbf{0}$. Thus $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

By inspection, $E = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

14. Row reduce the augmented matrix for $x_1 \begin{bmatrix} 1 \\ a \end{bmatrix} + x_2 \begin{bmatrix} a \\ a+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (*).

$$\begin{bmatrix} 1 & a & 0 \\ a & a+2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & a & 0 \\ 0 & a+2-a^2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 \\ 0 & (2-a)(1+a) & 0 \end{bmatrix}$$

The equation (*) has a nontrivial solution only when $(2 - a)(1 + a) = 0$. So the vectors are linearly independent for all a except $a = 2$ and $a = -1$.

15. **a.** If the three vectors are linearly independent, then a, c, and f must all be nonzero. (The converse is true, too.) Let A be the matrix whose columns are the three linearly independent vectors. Then

A must have three pivot columns. (See Exercise 30 in Section 1.7, or realize that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and so there can be no free variables in the system of equations.) Since A is 3×3, the pivot positions are exactly where a, c, and f are located.

b. The numbers $a, \ldots, f$ can have any values. Here's why. Denote the columns by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Observe that $\mathbf{v}_1$ is not the zero vector. Next, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ because the third entry of $\mathbf{v}_2$ is nonzero. Finally, $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because the fourth entry of $\mathbf{v}_3$ is nonzero. By Theorem 7 in Section 1.7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

16. Denote the columns from right to left by $\mathbf{v}_1, \ldots, \mathbf{v}_4$. The "first" vector $\mathbf{v}_1$ is nonzero, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ (because the third entry of $\mathbf{v}_2$ is nonzero), and $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ (because the second entry of $\mathbf{v}_3$ is nonzero). Finally, by looking at first entries in the vectors, $\mathbf{v}_4$ cannot be a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. By Theorem 7 in Section 1.7, the columns are linearly independent.

17. Here are two arguments. The first is a "direct" proof. The second is called a "proof by contradiction."

i. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set, $\mathbf{v}_1 \neq \mathbf{0}$. Also, Theorem 7 shows that $\mathbf{v}_2$ cannot be a multiple of $\mathbf{v}_1$, and $\mathbf{v}_3$ cannot be a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. By hypothesis, $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Thus, by Theorem 7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ cannot be a linearly dependent set and so must be linearly independent.

ii. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent. Then by Theorem 7, one of the vectors in the set is a linear combination of the preceding vectors. This vector cannot be $\mathbf{v}_4$ because $\mathbf{v}_4$ is *not* in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Also, none of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linear combinations of the preceding vectors, by Theorem 7. So the linear dependence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is impossible. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent.

18. Suppose that c_1 and c_2 are constants such that

$$c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{0} \qquad (*)$$

Then $(c_1 + c_2)\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, both $c_1 + c_2 = 0$ and $c_2 = 0$. It follows that both c_1 and c_2 in (*) must be zero, which shows that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2\}$ is linearly independent.

19. Let M be the line through the origin that is parallel to the line through $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Then $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ are both on M. So one of these two vectors is a multiple of the other, say $\mathbf{v}_2 - \mathbf{v}_1 = k(\mathbf{v}_3 - \mathbf{v}_1)$. This equation produces a linear dependence relation $(k - 1)\mathbf{v}_1 + \mathbf{v}_2 - k\mathbf{v}_3 = \mathbf{0}$.

A second solution: A parametric equation of the line is $\mathbf{x} = \mathbf{v}_1 + t(\mathbf{v}_2 - \mathbf{v}_1)$. Since $\mathbf{v}_3$ is on the line, there is some t_0 such that $\mathbf{v}_3 = \mathbf{v}_1 + t_0(\mathbf{v}_2 - \mathbf{v}_1) = (1 - t_0)\mathbf{v}_1 + t_0\mathbf{v}_2$. So $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

20. If $T(\mathbf{u}) = \mathbf{v}$, then since T is linear,

$$T(-\mathbf{u}) = T((-1)\mathbf{u}) = (-1)T(\mathbf{u}) = -\mathbf{v}.$$

21. Either compute $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, and $T(\mathbf{e}_3)$ to make the columns of A, or write the vectors vertically in the definition of T and fill in the entries of A by inspection:

$$A\mathbf{x} = \begin{bmatrix} ? & ? & ? \\ ? & A & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

22. By Theorem 12 in Section 1.9, the columns of A span $\mathbf{R}^3$. By Theorem 4 in Section 1.4, A has a pivot in each of its three rows. Since A has three columns, each column must be a pivot column. So the equation

$A\mathbf{x} = \mathbf{0}$ has no free variables, and the columns of A are linearly independent. By Theorem 12 in Section 1.9, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

23. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ implies that $\begin{matrix} 4a & - & 3b & = & 5 \\ 3a & + & 4b & = & 0 \end{matrix}$. Solve:

$$\begin{bmatrix} 4 & -3 & 5 \\ 3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & -3 & 5 \\ 0 & 25/4 & -15/4 \end{bmatrix} \sim \begin{bmatrix} 4 & -3 & 5 \\ 0 & 1 & -3/5 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 16/5 \\ 0 & 1 & -3/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4/5 \\ 0 & 1 & -3/5 \end{bmatrix}$$

Thus $a = 4/5$ and $b = -3/5$.

24. The matrix equation displayed gives the information $2a - 4b = 2\sqrt{5}$ and $4a + 2b = 0$. Solve for a and b:

$$\begin{bmatrix} 2 & -4 & 2\sqrt{5} \\ 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 2\sqrt{5} \\ 0 & 10 & -4\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & \sqrt{5} \\ 0 & 1 & -2/\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/\sqrt{5} \\ 0 & 1 & -2/\sqrt{5} \end{bmatrix}$$

So $a = 1/\sqrt{5},\ b = -2/\sqrt{5}$.

25. a. The vector lists the number of three-, two-, and one-bedroom apartments provided when x_1 floors of plan A are constructed.

b. $x_1\begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 3 \\ 9 \end{bmatrix}$

c. [M] Solve $x_1\begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 136 \end{bmatrix}$

$$\begin{bmatrix} 3 & 4 & 5 & 66 \\ 7 & 4 & 3 & 74 \\ 8 & 8 & 9 & 136 \end{bmatrix} \sim \cdots \begin{bmatrix} 1 & 0 & -1/2 & 2 \\ 0 & 1 & 13/8 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 & & & - & (1/2)x_3 & = & 2 \\ & & x_2 & + & (13/8)x_3 & = & 15 \\ & & & & 0 & = & 0 \end{matrix}$$

The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 + (1/2)x_3 \\ 15 - (13/8)x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1/2 \\ -13/8 \\ 1 \end{bmatrix}$$

However, the only feasible solutions must have whole numbers of floors for each plan. Thus, x_3 must be a multiple of 8, to avoid fractions. One solution, for $x_3 = 0$, is to use 2 floors of plan A and 15 floors of plan B. Another solution, for $x_3 = 8$, is to use 6 floors of plan A , 2 floors of plan B, and 8 floors of plan C. These are the only feasible solutions. A larger positive multiple of 8 for x_3 makes x_2 negative. A negative value for x_3, of course, is not feasible either.