2.2 SOLUTIONS

Notes: The text includes the matrix inversion algorithm at the end of the section because this topic is popular. Students like it because it is a simple mechanical procedure. However, I no longer cover it in my classes because technology is readily available to invert a matrix whenever needed, and class time is better spent on more useful topics such as partitioned matrices. The final subsection is independent of the inversion algorithm and is needed for Exercises 35 and 36.

Key Exercises: 8, 11–24, 35. (Actually, Exercise 8 is only helpful for some exercises in this section. Section 2.3 has a stronger result.) Exercises 23 and 24 are used in the proof of the Invertible Matrix Theorem (IMT) in Section 2.3, along with Exercises 23 and 24 in Section 2.1. I recommend letting students work on two or more of these four exercises before proceeding to Section 2.3. In this way students *participate* in the proof of the IMT rather than simply watch an instructor carry out the proof. Also, this activity will help students understand *why* the theorem is true.

1. $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}^{-1} = \frac{1}{32-30}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}^{-1} = \frac{1}{12-14}\begin{bmatrix} 4 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix}$

3. $\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}^{-1} = \frac{1}{-40-(-35)}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ -1.4 & -1.6 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}^{-1} = \frac{1}{-24-(-28)}\begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -7/4 & 3/4 \end{bmatrix}$

5. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and the solution is

$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$. Thus $x_1 = 7$ and $x_2 = -9$.

6. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 9 \\ 11 \end{bmatrix}$, and the solution is $\mathbf{x} = A^{-1}\mathbf{b}$. To compute this by hand, the arithmetic is simplified by keeping the fraction $1/\det(A)$ in front of the matrix for A^{-1}. (The *Study Guide* comments on this in its discussion of Exercise 7.) From Exercise 3,

$\mathbf{x} = A^{-1}\mathbf{b} = -\frac{1}{5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} -9 \\ 11 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -10 \\ 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$. Thus $x_1 = 2$ and $x_2 = -5$.

7. a. $\begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}^{-1} = \frac{1}{1 \cdot 12 - 2 \cdot 5}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}$ or $\begin{bmatrix} 6 & -1 \\ -2.5 & .5 \end{bmatrix}$

$\mathbf{x} = A^{-1}\mathbf{b}_1 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -18 \\ 8 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$. Similar calculations give

$A^{-1}\mathbf{b}_2 = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$, $A^{-1}\mathbf{b}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $A^{-1}\mathbf{b}_4 = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$.

b. $[A \;\; \mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3 \;\; \mathbf{b}_4] = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$

The solutions are $\begin{bmatrix} -9 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 11 \\ -5 \end{bmatrix}$, $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 13 \\ -5 \end{bmatrix}$, the same as in part (a).

Note: The *Study Guide* also discusses the number of arithmetic calculations for this Exercise 7, stating that when A is large, the method used in (b) is much faster than using A^{-1}.

8. Left-multiply each side of the equation $AD = I$ by A^{-1} to obtain

$A^{-1}AD = A^{-1}I$, $ID = A^{-1}$, and $D = A^{-1}$.

Parentheses are routinely suppressed because of the associative property of matrix multiplication.

9. a. True, by definition of *invertible.** **b.** False. See Theorem 6(b).

c. False. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $ab - cd = 1 - 0 \neq 0$, but Theorem 4 shows that this matrix is not invertible, because $ad - bc = 0$.

d. True. This follows from Theorem 5, which also says that the solution of $A\mathbf{x} = \mathbf{b}$ is unique, for each $\mathbf{b}$.

e. True, by the box just before Example 6.

10. a. False. The product matrix is invertible, but the product of inverses should be in the *reverse* order. See Theorem 6(b).

b. True, by Theorem 6(a). **c.** True, by Theorem 4.

d. True, by Theorem 7. **e.** False. The last part of Theorem 7 is misstated here.

11. (The proof can be modeled after the proof of Theorem 5.) The $n \times p$ matrix B is given (but is arbitrary). Since A is invertible, the matrix $A^{-1}B$ satisfies $AX = B$, because $A(A^{-1}B) = A\,A^{-1}B = IB = B$. To show this solution is unique, let X be any solution of $AX = B$. Then, left-multiplication of each side by A^{-1} shows that X must be $A^{-1}B$:

$$A^{-1}(AX) = A^{-1}B, \quad IX = A^{-1}B, \quad \text{and} \quad X = A^{-1}B.$$

12. If you assign this exercise, consider giving the following *Hint*: Use elementary matrices and imitate the proof of Theorem 7. The solution in the Instructor's Edition follows this hint. Here is another solution, based on the idea at the end of Section 2.2.

Write $B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$ and $X = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_p]$. By definition of matrix multiplication, $AX = [A\mathbf{u}_1 \ \cdots \ A\mathbf{u}_p]$. Thus, the equation $AX = B$ is equivalent to the p systems:

$$A\mathbf{u}_1 = \mathbf{b}_1, \ \ldots \ A\mathbf{u}_p = \mathbf{b}_p$$

Since A is the coefficient matrix in each system, these systems may be solved simultaneously, placing the augmented columns of these systems next to A to form $[A \ \ \mathbf{b}_1 \ \cdots \ \mathbf{b}_p] = [A \ \ B]$. Since A is invertible, the solutions $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are uniquely determined, and $[A \ \ \mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$ must row reduce to $[I \ \ \mathbf{u}_1 \ \cdots \ \mathbf{u}_p] = [I \ \ X]$. By Exercise 11, X is the unique solution $A^{-1}B$ of $AX = B$.

13. Left-multiply each side of the equation $AB = AC$ by A^{-1} to obtain

$$A^{-1}AB = A^{-1}AC, \quad IB = IC, \quad \text{and} \quad B = C.$$

This conclusion does not always follow when A is singular. Exercise 10 of Section 2.1 provides a counterexample.

14. Right-multiply each side of the equation $(B - C)D = 0$ by D^{-1} to obtain

$$(B - C)DD^{-1} = 0D^{-1}, \quad (B - C)I = 0, \quad B - C = 0, \quad \text{and} \quad B = C.$$

15. The box following Theorem 6 suggests what the inverse of ABC should be, namely, $C^{-1}B^{-1}A^{-1}$. To verify that this is correct, compute:

$$(ABC)\,C^{-1}B^{-1}A^{-1} = ABCC^{-1}B^{-1}A^{-1} = ABIB^{-1}A^{-1} = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$C^{-1}B^{-1}A^{-1}\,(ABC) = C^{-1}B^{-1}A^{-1}ABC = C^{-1}B^{-1}IBC = C^{-1}B^{-1}BC = C^{-1}IC = C^{-1}C = I$$

16. Let $C = AB$. Then $CB^{-1} = ABB^{-1}$, so $CB^{-1} = AI = A$. This shows that A is the product of invertible matrices and hence is invertible, by Theorem 6.

Note: The *Study Guide* warns against using the formula $(AB)^{-1} = B^{-1}A^{-1}$ here, because this formula can be used only when both A and B are already known to be invertible.

17. Right-multiply each side of $AB = BC$ by B^{-1}:

$$ABB^{-1} = BCB^{-1}, \quad AI = BCB^{-1}, \quad A = BCB^{-1}.$$

18. Left-multiply each side of $A = PBP^{-1}$ by P^{-1}:

$$P^{-1}A = P^{-1}PBP^{-1}, \quad P^{-1}A = IBP^{-1}, \quad P^{-1}A = BP^{-1}$$

Then right-multiply each side of the result by P:

$$P^{-1}AP = BP^{-1}P, \quad P^{-1}AP = BI, \quad P^{-1}AP = B$$

19. Unlike Exercise 17, this exercise asks two things, "Does a solution exist and what is it?" First, find what the solution must be, if it exists. That is, suppose X satisfies the equation $C^{-1}(A + X)B^{-1} = I$. Left-multiply each side by C, and then right-multiply each side by B:

$$CC^{-1}(A + X)B^{-1} = CI, \quad I(A + X)B^{-1} = C, \quad (A + X)B^{-1}B = CB, \quad (A + X)I = CB$$

Expand the left side and then subtract A from both sides:

$$AI + XI = CB, \quad A + X = CB, \quad X = CB - A$$

If a solution exists, it must be $CB - A$. To show that $CB - A$ really *is* a solution, substitute it for X:

$$C^{-1}[A + (CB - A)]B^{-1} = C^{-1}[CB]B^{-1} = C^{-1}CBB^{-1} = II = I.$$

Note: The *Study Guide* suggests that students ask their instructor about how many details to include in their proofs. After some practice with algebra, an expression such as $CC^{-1}(A + X)B^{-1}$ could be simplified directly to $(A + X)B^{-1}$ without first replacing CC^{-1} by I. However, you may wish this detail to be included in the homework for this section.

20. **a.** Left-multiply both sides of $(A - AX)^{-1} = X^{-1}B$ by X to see that B is invertible because it is the product of invertible matrices.

b. Invert both sides of the original equation and use Theorem 6 about the inverse of a product (which applies because X^{-1} and B are invertible):

$$A - AX = (X^{-1}B)^{-1} = B^{-1}(X^{-1})^{-1} = B^{-1}X$$

Then $A = AX + B^{-1}X = (A + B^{-1})X$. The product $(A + B^{-1})X$ is invertible because A is invertible. Since X is known to be invertible, so is the other factor, $A + B^{-1}$, by Exercise 16 or by an argument similar to part (a). Finally,

$$(A + B^{-1})^{-1}A = (A + B^{-1})^{-1}(A + B^{-1})X = X$$

Note: This exercise is difficult. The algebra is not trivial, and at this point in the course, most students will not recognize the need to verify that a matrix is invertible.

21. Suppose A is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{0}$ has only one solution, namely, the zero solution. This means that the columns of A are linearly independent, by a remark in Section 1.7.

22. Suppose A is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{b}$ has a solution (in fact, a unique solution) for each $\mathbf{b}$. By Theorem 4 in Section 1.4, the columns of A span $\mathbf{R}^n$.

23. Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then there are no free variables in this equation, and so A has n pivot columns. Since A is *square* and the n pivot positions must be in different rows, the pivots in an echelon form of A must be on the main diagonal. Hence A is row equivalent to the $n \times n$ identity matrix.

24. If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^n$, then A has a pivot position in each row, by Theorem 4 in Section 1.4. Since A is square, the pivots must be on the diagonal of A. It follows that A is row equivalent to I_n. By Theorem 7, A is invertible.

25. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc = 0$. If $a = b = 0$, then examine $\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ This has the solution $\mathbf{x}_1 = \begin{bmatrix} d \\ -c \end{bmatrix}$. This solution is nonzero, except when $a = b = c = d$. In that case, however, A is the zero matrix, and $A\mathbf{x} = \mathbf{0}$ for *every* vector $\mathbf{x}$. Finally, if a and b are not both zero, set $\mathbf{x}_2 = \begin{bmatrix} -b \\ a \end{bmatrix}$. Then

$A\mathbf{x}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} -ab + ba \\ -cb + da \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, because $-cb + da = 0$. Thus, $\mathbf{x}_2$ is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

So, in all cases, the equation $A\mathbf{x} = \mathbf{0}$ has more than one solution. This is impossible when A is invertible (by Theorem 5), so A is *not* invertible.

26. $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da - bc & 0 \\ 0 & -cb + ad \end{bmatrix}$. Divide both sides by $ad - bc$ to get $CA = I$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & -cb + da \end{bmatrix}.$$

Divide both sides by $ad - bc$. The right side is I. The left side is AC, because

$$\frac{1}{ad - bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = AC$$

27. a. Interchange A and B in equation (1) after Example 6 in Section 2.1: $\text{row}_i(BA) = \text{row}_i(B)\cdot A$. Then replace B by the identity matrix: $\text{row}_i(A) = \text{row}_i(IA) = \text{row}_i(I)\cdot A$.

b. Using part (a), when rows 1 and 2 of A are interchanged, write the result as

$$\begin{bmatrix} \text{row}_2(A) \\ \text{row}_1(A) \\ \text{row}_3(A) \end{bmatrix} = \begin{bmatrix} \text{row}_2(I)\cdot A \\ \text{row}_1(I)\cdot A \\ \text{row}_3(I)\cdot A \end{bmatrix} = \begin{bmatrix} \text{row}_2(I) \\ \text{row}_1(I) \\ \text{row}_3(I) \end{bmatrix} A = EA \qquad (*)$$

Here, E is obtained by interchanging rows 1 and 2 of I. The second equality in (*) is a consequence of the fact that $\text{row}_i(EA) = \text{row}_i(E)\cdot A$.

c. Using part (a), when row 3 of A is multiplied by 5, write the result as

$$\begin{bmatrix} \text{row}_1(A) \\ \text{row}_2(A) \\ 5\cdot\text{row}_3(A) \end{bmatrix} = \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ 5\cdot\text{row}_3(I)\cdot A \end{bmatrix} = \begin{bmatrix} \text{row}_1(I) \\ \text{row}_2(I) \\ 5\cdot\text{row}_3(I) \end{bmatrix} A = EA$$

Here, E is obtained by multiplying row 3 of I by 5.

28. When row 3 of A is replaced by $\text{row}_3(A) - 4\cdot\text{row}_1(A)$, write the result as

$$\begin{bmatrix} \text{row}_1(A) \\ \text{row}_2(A) \\ \text{row}_3(A) - 4\cdot\text{row}_1(A) \end{bmatrix} = \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ \text{row}_3(I)\cdot A - 4\cdot\text{row}_1(I)\cdot A \end{bmatrix}$$

$$= \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ [\text{row}_3(I) - 4\cdot \text{row}_1(I)]\cdot A \end{bmatrix} = \begin{bmatrix} \text{row}_1(I) \\ \text{row}_2(I) \\ \text{row}_3(I) - 4\cdot \text{row}_1(I) \end{bmatrix} A = EA$$

Here, E is obtained by replacing $\text{row}_3(I)$ by $\text{row}_3(I) - 4\cdot\text{row}_1(I)$.

29. $[A \quad I] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 4 & -1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$

30. $[A \quad I] = \begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 0 & -1 & -4/5 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 0 & 1 & 4/5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7/5 & 2 \\ 0 & 1 & 4/5 & -1 \end{bmatrix}$. $A^{-1} = \begin{bmatrix} -7/5 & 2 \\ 4/5 & -1 \end{bmatrix}$

31. $[A \quad I] = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix}$. $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$

32. $[A \quad I] = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{bmatrix}$. The matrix A is not invertible.

33. Let $B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$, and for $j = 1, \ldots, n$, let $\mathbf{a}_j$, $\mathbf{b}_j$, and $\mathbf{e}_j$ denote the jth columns of A, B, and I, respectively. Note that for $j = 1, \ldots, n-1$, $\mathbf{a}_j - \mathbf{a}_{j+1} = \mathbf{e}_j$ (because $\mathbf{a}_j$ and $\mathbf{a}_{j+1}$ have the same entries except for the jth row), $\mathbf{b}_j = \mathbf{e}_j - \mathbf{e}_{j+1}$ and $\mathbf{a}_n = \mathbf{b}_n = \mathbf{e}_n$.

To show that $AB = I$, it suffices to show that $A\mathbf{b}_j = \mathbf{e}_j$ for each j. For $j = 1, \ldots, n-1$,

$$A\mathbf{b}_j = A(\mathbf{e}_j - \mathbf{e}_{j+1}) = A\mathbf{e}_j - A\mathbf{e}_{j+1} = \mathbf{a}_j - \mathbf{a}_{j+1} = \mathbf{e}_j$$

and $A\mathbf{b}_n = A\mathbf{e}_n = \mathbf{a}_n = \mathbf{e}_n$. Next, observe that $\mathbf{a}_j = \mathbf{e}_j + \cdots + \mathbf{e}_n$ for each j. Thus,

$$B\mathbf{a}_j = B(\mathbf{e}_j + \cdots + \mathbf{e}_n) = \mathbf{b}_j + \cdots + \mathbf{b}_n$$
$$= (\mathbf{e}_j - \mathbf{e}_{j+1}) + (\mathbf{e}_{j+1} - \mathbf{e}_{j+2}) + \cdots + (\mathbf{e}_{n-1} - \mathbf{e}_n) + \mathbf{e}_n = \mathbf{e}_j$$

This proves that $BA = I$. Combined with the first part, this proves that $B = A^{-1}$.

Note: Students who do this problem and then do the corresponding exercise in Section 2.4 will appreciate the Invertible Matrix Theorem, partitioned matrix notation, and the power of a proof by induction.

34. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1/2 & 1/2 & 0 & & \\ 0 & -1/3 & 1/3 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & -1/n & 1/n \end{bmatrix}$$

and for $j = 1, \ldots, n$, let $\mathbf{a}_j$, $\mathbf{b}_j$, and $\mathbf{e}_j$ denote the jth columns of A, B, and I, respectively. Note that for $j = 1, \ldots, n-1$, $\mathbf{a}_j = j(\mathbf{e}_j + \cdots + \mathbf{e}_n)$, $\mathbf{b}_j = \frac{1}{j}\mathbf{e}_j - \frac{1}{j+1}\mathbf{e}_{j+1}$, and $\mathbf{b}_n = \frac{1}{n}\mathbf{e}_n$.

To show that $AB = I$, it suffices to show that $A\mathbf{b}_j = \mathbf{e}_j$ for each j. For $j = 1, \ldots, n-1$,

$$A\mathbf{b}_j = A\left(\frac{1}{j}\mathbf{e}_j - \frac{1}{j+1}\mathbf{e}_{j+1}\right) = \frac{1}{j}\mathbf{a}_j - \frac{1}{j+1}\mathbf{a}_{j+1}$$
$$= (\mathbf{e}_j + \cdots + \mathbf{e}_n) - (\mathbf{e}_{j+1} + \cdots + \mathbf{e}_n) = \mathbf{e}_j$$

Also, $A\mathbf{b}_n = A\left(\frac{1}{n}\mathbf{e}_n\right) = \frac{1}{n}\mathbf{a}_n = \mathbf{e}_n$. Finally, for $j = 1, \ldots, n$, the sum $\mathbf{b}_j + \cdots + \mathbf{b}_n$ is a "telescoping sum" whose value is $\frac{1}{j}\mathbf{e}_j$. Thus,

$$B\mathbf{a}_j = j(B\mathbf{e}_j + \cdots + B\mathbf{e}_n) = j(\mathbf{b}_j + \cdots + \mathbf{b}_n) = j\left(\frac{1}{j}\mathbf{e}_j\right) = \mathbf{e}_j$$

which proves that $BA = I$. Combined with the first part, this proves that $B = A^{-1}$.

Note: If you assign Exercise 34, you may wish to supply a hint using the notation from Exercise 33: Express each column of A in terms of the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the identity matrix. Do the same for B.

35. Row reduce $[A \;\; \mathbf{e}_3]$:

$$\begin{bmatrix} -2 & -7 & -9 & 0 \\ 2 & 5 & 6 & 0 \\ 1 & 3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 2 & 5 & 6 & 0 \\ -2 & -7 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & -15 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -15 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Answer: The third column of A^{-1} is $\begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$.

36. **[M]** Write $B = [A \quad F]$, where F consists of the last two columns of I_3, and row reduce:

$$B = \begin{bmatrix} -25 & -9 & -27 & 0 & 0 \\ 546 & 180 & 537 & 1 & 0 \\ 154 & 50 & 149 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3/2 & -9/2 \\ 0 & 1 & 0 & -433/6 & 439/2 \\ 0 & 0 & 1 & 68/3 & -69 \end{bmatrix}$$

The last two columns of A^{-1} are $\begin{bmatrix} 1.5000 & -4.5000 \\ -72.1667 & 219.5000 \\ 22.6667 & -69.0000 \end{bmatrix}$

37. There are many possibilities for C, but $C = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ is the only one whose entries are 1, –1, and 0.

With only three possibilities for each entry, the construction of C can be done by trial and error. This is probably faster than setting up a system of 4 equations in 6 unknowns. The fact that A cannot be invertible follows from Exercise 25 in Section 2.1, because A is not square.

38. Write $AD = A[\mathbf{d}_1 \quad \mathbf{d}_2] = [A\mathbf{d}_1 \quad A\mathbf{d}_2]$. The structure of A shows that $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$.

[There are 25 possibilities for D if entries of D are allowed to be 1, –1, and 0.] There is *no* 4×2 matrix C such that $CA = I_4$. If this were true, then $CA\mathbf{x}$ would equal $\mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^4$. This cannot happen because the columns of A are linearly dependent and so $A\mathbf{x} = \mathbf{0}$ for some nonzero vector $\mathbf{x}$. For such an $\mathbf{x}$, $CA\mathbf{x} = C(\mathbf{0}) = \mathbf{0}$. An alternate justification would be to cite Exercise 23 or 25 in Section 2.1.

39. $\mathbf{y} = D\mathbf{f} = \begin{bmatrix} .005 & .002 & .001 \\ .002 & .004 & .002 \\ .001 & .002 & .005 \end{bmatrix}\begin{bmatrix} 30 \\ 50 \\ 20 \end{bmatrix} = \begin{bmatrix} .27 \\ .30 \\ .23 \end{bmatrix}$. The deflections are .27 in., .30 in., and .23 in. at points 1, 2, and 3, respectively.

40. **[M]** The *stiffness matrix* is D^{-1}. Use an "inverse" command to produce

$$D^{-1} = 125\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

To find the forces (in pounds) required to produce a deflection of .04 cm at point 3, most students will use technology to solve $D\mathbf{f} = (0, 0, .04)$ and obtain (0, –5, 10).

Here is another method, based on the idea suggested in Exercise 42. The first column of D^{-1} lists the forces required to produce a deflection of 1 in. at point 1 (with zero deflection at the other points). Since the transformation $\mathbf{y} \mapsto D^{-1}\mathbf{y}$ is linear, the forces required to produce a deflection of .04 cm at point 3 is given by .04 times the third column of D^{-1}, namely (.04)(125) times (0, –1, 2), or (0, –5, 10) pounds.

41. To determine the forces that produce a deflections of .08, .12, .16, and .12 cm at the four points on the beam, use technology to solve $D\mathbf{f} = \mathbf{y}$, where $\mathbf{y} = (.08, .12, .16, .12)$. The forces at the four points are 12, 1.5, 21.5, and 12 newtons, respectively.

42. **[M]** To determine the forces that produce a deflection of .240 cm at the second point on the beam, use technology to solve $D\mathbf{f} = \mathbf{y}$, where $\mathbf{y} = (0, .24, 0, 0)$. The forces at the four points are –104, 167, –113, and 56.0 newtons, respectively (to three significant digits). These forces are .24 times the entries in the second column of D^{-1}. *Reason*: The transformation $\mathbf{y} \mapsto D^{-1}\mathbf{y}$ is linear, so the forces required to produce a deflection of .24 cm at the second point are .24 times the forces required to produce a deflection of 1 cm at the second point. These forces are listed in the second column of D^{-1}.

Another possible discussion: The solution of $D\mathbf{x} = (0, 1, 0, 0)$ is the second column of D^{-1}. Multiply both sides of this equation by .24 to obtain $D(.24\mathbf{x}) = (0, .24, 0, 0)$. So $.24\mathbf{x}$ is the solution of $D\mathbf{f} = (0, .24, 0, 0)$. (The argument uses linearity, but students may not mention this.)

Note: The *Study Guide* suggests using **`gauss`**, **`swap`**, **`bgauss`**, and **`scale`** to reduce $[A \quad I]$, because I prefer to postpone the use of **`ref`** (or **`rref`**) until later. If you wish to introduce **`ref`** now, see the *Study Guide*'s technology notes for Sections 2.8 or 4.3. (Recall that Sections 2.8 and 2.9 are only covered when an instructor plans to skip Chapter 4 and get quickly to eigenvalues.)