4.2 SOLUTIONS

Notes: This section provides a review of Chapter 1 using the new terminology. Linear tranformations are introduced quickly since students are already comfortable with the idea from $\mathbb{R}^n$. The key exercises are 17–26, which are straightforward but help to solidify the notions of null spaces and column spaces. Exercises 30–36 deal with the kernel and range of a linear transformation and are progressively more advanced theoretically. The idea in Exercises 7–14 is for the student to use Theorems 1, 2, or 3 to determine whether a given set is a subspace.

1. One calculates that

$$A\mathbf{w} = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

so $\mathbf{w}$ is in Nul A.

2. One calculates that

$$A\mathbf{w} = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

so $\mathbf{w}$ is in Nul A.

3. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix},$$

the general solution is $x_1 = 7x_3 - 6x_4$, $x_2 = -4x_3 + 2x_4$, with x_3 and x_4 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix},$$

and a spanning set for Nul A is

$$\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

4. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & -6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

the general solution is $x_1 = 6x_2$, $x_3 = 0$, with x_2 and x_4 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and a spanning set for Nul A is

$$\left\{ \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

5. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

the general solution is $x_1 = 2x_2 - 4x_4$, $x_3 = 9x_4$, $x_5 = 0$, with x_2 and x_4 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix},$$

and a spanning set for Nul A is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

6. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 & -8 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the general solution is $x_1 = -6x_3 + 8x_4 - x_5$, $x_2 = 2x_3 - x_4$, with x_3, x_4, and x_5 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and a spanning set for Nul A is

$$\left\{ \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

7. The set W is a subset of $\mathbb{R}^3$. If W were a vector space (under the standard operations in $\mathbb{R}^3$), then it would be a subspace of $\mathbb{R}^3$. But W is not a subspace of $\mathbb{R}^3$ since the zero vector is not in W. Thus W is not a vector space.

8. The set W is a subset of $\mathbb{R}^3$. If W were a vector space (under the standard operations in $\mathbb{R}^3$), then it would be a subspace of $\mathbb{R}^3$. But W is not a subspace of $\mathbb{R}^3$ since the zero vector is not in W. Thus W is not a vector space.

9. The set W is the set of all solutions to the homogeneous system of equations $a-2b-4c=0$, $2a-c-3d=0$. Thus $W=\text{Nul}\,A$, where $A=\begin{bmatrix}1 & -2 & -4 & 0\\ 2 & 0 & -1 & -3\end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^4$ by Theorem 2, and is a vector space.

10. The set W is the set of all solutions to the homogeneous system of equations $a+3b-c=0$, $a+b+c-d=0$. Thus $W=\text{Nul}\,A$, where $A=\begin{bmatrix}1 & 3 & -1 & 0\\ 1 & 1 & 1 & -1\end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^4$ by Theorem 2, and is a vector space.

11. The set W is a subset of $\mathbb{R}^4$. If W were a vector space (under the standard operations in $\mathbb{R}^4$), then it would be a subspace of $\mathbb{R}^4$. But W is not a subspace of $\mathbb{R}^4$ since the zero vector is not in W. Thus W is not a vector space.

12. The set W is a subset of $\mathbb{R}^4$. If W were a vector space (under the standard operations in $\mathbb{R}^4$), then it would be a subspace of $\mathbb{R}^4$. But W is not a subspace of $\mathbb{R}^4$ since the zero vector is not in W. Thus W is not a vector space.

13. An element $\mathbf{w}$ on W may be written as

$$\mathbf{w}=c\begin{bmatrix}1\\0\\1\end{bmatrix}+d\begin{bmatrix}-6\\1\\0\end{bmatrix}=\begin{bmatrix}1 & -6\\0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}$$

where c and d are any real numbers. So $W=\text{Col}\,A$ where $A=\begin{bmatrix}1 & -6\\0 & 1\\1 & 0\end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^3$ by Theorem 3, and is a vector space.

14. An element $\mathbf{w}$ on W may be written as

$$\mathbf{w}=a\begin{bmatrix}-1\\1\\3\end{bmatrix}+b\begin{bmatrix}2\\-2\\-6\end{bmatrix}=\begin{bmatrix}-1 & 2\\1 & -2\\3 & -6\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$$

where a and b are any real numbers. So $W=\text{Col}\,A$ where $A=\begin{bmatrix}-1 & 2\\1 & -2\\3 & -6\end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^3$ by Theorem 3, and is a vector space.

15. An element in this set may be written as

$$r\begin{bmatrix}0\\1\\4\\3\end{bmatrix}+s\begin{bmatrix}2\\1\\1\\-1\end{bmatrix}+t\begin{bmatrix}3\\-2\\0\\-1\end{bmatrix}=\begin{bmatrix}0&2&3\\1&1&-2\\4&1&0\\3&-1&-1\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix}$$

where r, s and t are any real numbers. So the set is Col A where $A=\begin{bmatrix}0&2&3\\1&1&-2\\4&1&0\\3&-1&-1\end{bmatrix}$.

16. An element in this set may be written as

$$b\begin{bmatrix}1\\2\\0\\0\end{bmatrix}+c\begin{bmatrix}-1\\1\\5\\0\end{bmatrix}+d\begin{bmatrix}0\\1\\-4\\1\end{bmatrix}=\begin{bmatrix}1&-1&0\\2&1&1\\0&5&-4\\0&0&1\end{bmatrix}\begin{bmatrix}b\\c\\d\end{bmatrix}$$

where b, c and d are any real numbers. So the set is Col A where $A=\begin{bmatrix}1&-1&0\\2&1&1\\0&5&-4\\0&0&1\end{bmatrix}$.

17. The matrix A is a 4×2 matrix. Thus
(a) Nul A is a subspace of $\mathbb{R}^2$, and
(b) Col A is a subspace of $\mathbb{R}^4$.

18. The matrix A is a 4×3 matrix. Thus
(a) Nul A is a subspace of $\mathbb{R}^3$, and
(b) Col A is a subspace of $\mathbb{R}^4$.

19. The matrix A is a 2×5 matrix. Thus
(a) Nul A is a subspace of $\mathbb{R}^5$, and
(b) Col A is a subspace of $\mathbb{R}^2$.

20. The matrix A is a 1×5 matrix. Thus
(a) Nul A is a subspace of $\mathbb{R}^5$, and
(b) Col A is a subspace of $\mathbb{R}^1=\mathbb{R}$.

21. Either column of A is a nonzero vector in Col A. To find a nonzero vector in Nul A, find the general solution of $A\mathbf{x}=\mathbf{0}$ in terms of the free variables. Since

$$[A\ \ \mathbf{0}]\sim\begin{bmatrix}1&-3&0\\0&0&0\\0&0&0\\0&0&0\end{bmatrix},$$

the general solution is $x_1 = 3x_2$, with x_2 free. Letting x_2 be a nonzero value (say $x_2 = 1$) gives the nonzero vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

which is in Nul A.

22. Any column of A is a nonzero vector in Col A. To find a nonzero vector in Nul A, find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix},$$

the general solution is $x_1 = 7x_3 - 6x_4$, $x_2 = -4x_3 + 2x_4$, with x_3 and x_4 free. Letting x_3 and x_4 be nonzero values (say $x_3 = x_4 = 1$) gives the nonzero vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

which is in Nul A.

23. Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w} \end{bmatrix}$. Since

$$\begin{bmatrix} A & \mathbf{w} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1/3 \\ 0 & 0 & 0 \end{bmatrix},$$

the system is consistent and $\mathbf{w}$ is in Col A. Also, since

$$A\mathbf{w} = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mathbf{w}$ is in Nul A.

24. Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w} \end{bmatrix}$. Since

$$\begin{bmatrix} A & \mathbf{w} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1/2 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the system is consistent and $\mathbf{w}$ is in Col A. Also, since

$$A\mathbf{w} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{w}$ is in Nul A.

25. **a**. True. See the definition before Example 1.

b. False. See Theorem 2.

c. True. See the remark just before Example 4.

d. False. The equation $A\mathbf{x} = \mathbf{b}$ must be consistent *for every* $\mathbf{b}$. See #7 in the table on page 226.

e. True. See Figure 2.

f. True. See the remark after Theorem 3.

26. **a.** True. See Theorem 2.

b. True. See Theorem 3.

c. False. See the box after Theorem 3.

d. True. See the paragraph after the definition of a linear transformation.

e. True. See Figure 2.

f. True. See the paragraph before Example 8.

27. Let A be the coefficient matrix of the given homogeneous system of equations. Since $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{x}$ is in Nul$A$. Since Nul$A$ is a subspace of $\mathbb{R}^3$, it is closed under scalar multiplication. Thus $10\mathbf{x} = \begin{bmatrix} 30 \\ 20 \\ -10 \end{bmatrix}$ is also in Nul$A$, and $x_1 = 30$, $x_2 = 20$, $x_3 = -10$ is also a solution to the system of equations.

28. Let A be the coefficient matrix of the given systems of equations. Since the first system has a solution, the constant vector $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$ is in Col$A$. Since Col A is a subspace of $\mathbb{R}^3$, it is closed under scalar multiplication. Thus $5\mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix}$ is also in Col A, and the second system of equations must thus have a solution.

29. **a.** Since $A\mathbf{0} = \mathbf{0}$, the zero vector is in Col A.

b. Since $A\mathbf{x} + A\mathbf{w} = A(\mathbf{x} + \mathbf{w}), A\mathbf{x} + A\mathbf{w}$ is in Col A.

c. Since $c(A\mathbf{x}) = A(c\mathbf{x}), cA\mathbf{x}$ is in Col A.

30. Since $T(\mathbf{0}_V) = \mathbf{0}_W$, the zero vector $\mathbf{0}_W$ of W is in the range of T. Let $T(\mathbf{x})$ and $T(\mathbf{w})$ be typical elements in the range of T. Then since $T(\mathbf{x}) + T(\mathbf{w}) = T(\mathbf{x} + \mathbf{w}), T(\mathbf{x}) + T(\mathbf{w})$ is in the range of T and the range of T is closed under vector addition. Let c be any scalar. Then since $cT(\mathbf{x}) = T(c\mathbf{x}), cT(\mathbf{x})$ is in the range of T and the range of T is closed under scalar multiplication. Hence the range of T is a subspace of W.

31. **a.** Let $\mathbf{p}$ and $\mathbf{q}$ be arbitary polynomials in $\mathbb{P}_2$, and let c be any scalar. Then

$$T(\mathbf{p} + \mathbf{q}) = \begin{bmatrix} (\mathbf{p} + \mathbf{q})(0) \\ (\mathbf{p} + \mathbf{q})(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) + \mathbf{q}(0) \\ \mathbf{p}(1) + \mathbf{q}(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(0) \\ \mathbf{q}(1) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q})$$

and

$$T(c\mathbf{p}) = \begin{bmatrix} (c\mathbf{p})(0) \\ (c\mathbf{p})(1) \end{bmatrix} = c\begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = cT(\mathbf{p})$$

so T is a linear transformation.

b. Any quadratic polynomial $\mathbf{q}$ for which $\mathbf{q}(0)=0$ and $\mathbf{q}(1)=0$ will be in the kernel of T. The polynomial $\mathbf{q}$ must then be a multiple of $\mathbf{p}(t)=t(t-1)$. Given any vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$, the polynomial $\mathbf{p}=x_1+(x_2-x_1)t$ has $\mathbf{p}(0)=x_1$ and $\mathbf{p}(1)=x_2$. Thus the range of T is all of $\mathbb{R}^2$.

32. Any quadratic polynomial $\mathbf{q}$ for which $\mathbf{q}(0)=0$ will be in the kernel of T. The polynomial $\mathbf{q}$ must then be $\mathbf{q}=at+bt^2$. Thus the polynomials $\mathbf{p}_1(t)=t$ and $\mathbf{p}_2(t)=t^2$ span the kernel of T. If a vector is in the range of T, it must be of the form $\begin{bmatrix} a \\ a \end{bmatrix}$. If a vector is of this form, it is the image of the polynomial $\mathbf{p}(t)=a$ in $\mathbb{P}_2$. Thus the range of T is $\left\{\begin{bmatrix} a \\ a \end{bmatrix} : a \text{ real}\right\}$.

33. a. For any A and B in $M_{2\times2}$ and for any scalar c,

$$T(A+B)=(A+B)+(A+B)^T=A+B+A^T+B^T=(A+A^T)+(B+B^T)=T(A)+T(B)$$

and

$$T(cA)=(cA)^T=c(A^T)=cT(A)$$

so T is a linear transformation.

b. Let B be an element of $M_{2\times2}$ with $B^T=B$, and let $A=\frac{1}{2}B$. Then

$$T(A)=A+A^T=\frac{1}{2}B+(\frac{1}{2}B)^T=\frac{1}{2}B+\frac{1}{2}B^T=\frac{1}{2}B+\frac{1}{2}B=B$$

c. Part b. showed that the range of T contains the set of all B in $M_{2\times2}$ with $B^T=B$. It must also be shown that any B in the range of T has this property. Let B be in the range of T. Then $B=T(A)$ for some A in $M_{2\times2}$. Then $B=A+A^T$, and

$$B^T=(A+A^T)^T=A^T+(A^T)^T=A^T+A=A+A^T=B$$

so B has the property that $B^T=B$.

d. Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be in the kernel of T. Then $T(A)=A+A^T=0$, so

$$A+A^T=\begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} a & c \\ b & d \end{bmatrix}=\begin{bmatrix} 2a & c+b \\ b+c & 2d \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving it is found that $a=d=0$ and $c=-b$. Thus the kernel of T is $\left\{\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \text{ real}\right\}$.

34. Let $\mathbf{f}$ and $\mathbf{g}$ be any elements in $C[0, 1]$ and let c be any scalar. Then $T(\mathbf{f})$ is the antiderivative $\mathbf{F}$ of $\mathbf{f}$ with $\mathbf{F}(0)=0$ and $T(\mathbf{g})$ is the antiderivative $\mathbf{G}$ of $\mathbf{g}$ with $\mathbf{G}(0)=0$. By the rules for antidifferentiation $\mathbf{F}+\mathbf{G}$ will be an antiderivative of $\mathbf{f}+\mathbf{g}$, and $(\mathbf{F}+\mathbf{G})(0)=\mathbf{F}(0)+\mathbf{G}(0)=0+0=0$. Thus $T(\mathbf{f}+\mathbf{g})=T(\mathbf{f})+T(\mathbf{g})$. Likewise $c\mathbf{F}$ will be an antiderivative of $c\mathbf{f}$, and $(c\mathbf{F})(0)=c\mathbf{F}(0)=c0=0$. Thus $T(c\mathbf{f})=cT(\mathbf{f})$, and T is a linear transformation. To find the kernel of T, we must find all functions f in $C[0,1]$ with antiderivative equal to the zero function. The only function with this property is the zero function $\mathbf{0}$, so the kernel of T is $\{\mathbf{0}\}$.

35. Since U is a subspace of V, $\mathbf{0}_V$ is in U. Since T is linear, $T(\mathbf{0}_V)=\mathbf{0}_W$. So $\mathbf{0}_W$ is in $T(U)$. Let $T(\mathbf{x})$ and $T(\mathbf{y})$ be typical elements in $T(U)$. Then $\mathbf{x}$ and $\mathbf{y}$ are in U, and since U is a subspace of V, $\mathbf{x}+\mathbf{y}$ is also in U. Since T is linear, $T(\mathbf{x})+T(\mathbf{y})=T(\mathbf{x}+\mathbf{y})$. So $T(\mathbf{x})+T(\mathbf{y})$ is in $T(U)$, and $T(U)$ is closed under vector addition. Let c be any scalar. Then since $\mathbf{x}$ is in U and U is a subspace of V, $c\mathbf{x}$ is in U. Since T is linear, $T(c\mathbf{x})=cT(\mathbf{x})$ and $cT(\mathbf{x})$ is in $T(U)$. Thus $T(U)$ is closed under scalar multiplication, and $T(U)$ is a subspace of W.

36. Since Z is a subspace of W, $\mathbf{0}_W$ is in Z. Since T is linear, $T(\mathbf{0}_V)=\mathbf{0}_W$. So $\mathbf{0}_V$ is in U. Let $\mathbf{x}$ and $\mathbf{y}$ be typical elements in U. Then $T(\mathbf{x})$ and $T(\mathbf{y})$ are in Z, and since Z is a subspace of W, $T(\mathbf{x})+T(\mathbf{y})$ is also in Z. Since T is linear, $T(\mathbf{x})+T(\mathbf{y})=T(\mathbf{x}+\mathbf{y})$. So $T(\mathbf{x}+\mathbf{y})$ is in Z, and $\mathbf{x}+\mathbf{y}$ is in U. Thus U is closed under vector addition. Let c be any scalar. Then since $\mathbf{x}$ is in U, $T(\mathbf{x})$ is in Z. Since Z is a subspace of W, $cT(\mathbf{x})$ is also in Z. Since T is linear, $cT(\mathbf{x})=T(c\mathbf{x})$ and $T(c\mathbf{x})$ is in $T(U)$. Thus $c\mathbf{x}$ is in U and U is closed under scalar multiplication. Hence U is a subspace of V.

37. **[M]** Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w} \end{bmatrix}$. Since

$$\begin{bmatrix} A & \mathbf{w} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/95 & 1/95 \\ 0 & 1 & 0 & 39/19 & -20/19 \\ 0 & 0 & 1 & 267/95 & -172/95 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the system is consistent and $\mathbf{w}$ is in Col$A$. Also, since

$$A\mathbf{w} = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{w}$ is not in Nul$A$.

38. **[M]** Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w} \end{bmatrix}$. Since

$$\begin{bmatrix} A & \mathbf{w} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the system is consistent and $\mathbf{w}$ is in Col$A$. Also, since

$$A\mathbf{w} = \begin{bmatrix} -8 & 5 & -2 & 0 \\ -5 & 2 & 1 & -2 \\ 10 & -8 & 6 & -3 \\ 3 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{w}$ is in Nul$A$.

39. **[M]**

a. To show that $\mathbf{a}_3$ and $\mathbf{a}_5$ are in the column space of B, we can row reduce the matrices $\begin{bmatrix} B & \mathbf{a}_3 \end{bmatrix}$ and $\begin{bmatrix} B & \mathbf{a}_3 \end{bmatrix}$:

$$\begin{bmatrix} B & \mathbf{a}_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} B & \mathbf{a}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 10/3 \\ 0 & 1 & 0 & -26/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since both these systems are consistent, $\mathbf{a}_3$ and $\mathbf{a}_5$ are in the column space of B. Notice that the same conclusions can be drawn by observing the reduced row echelon form for A:

$$A \sim \begin{bmatrix} 1 & 0 & 1/3 & 0 & 10/3 \\ 0 & 1 & 1/3 & 0 & -26/3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables by using the reduced row echelon form of A given above: $x_1 = (-1/3)x_3 - (10/3)x_5$, $x_2 = (-1/3)x_3 + (26/3)x_5$, $x_4 = 4x_5$ with x_3 and x_5 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -10/3 \\ 26/3 \\ 0 \\ 4 \\ 1 \end{bmatrix},$$

and a spanning set for Nul A is

$$\left\{ \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10/3 \\ 26/3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

c. The reduced row echelon form of A shows that the columns of A are linearly dependent and do not span $\mathbb{R}^4$. Thus by Theorem 12 in Section 1.9, T is neither one-to-one nor onto.

40. **[M]** Since the line lies both in $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and in $K = \text{Span}\{\mathbf{v}_3, \mathbf{v}_4\}$, $\mathbf{w}$ can be written both as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ and $c_3\mathbf{v}_3 + c_4\mathbf{v}_4$. To find $\mathbf{w}$ we must find the c_j's which solve $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 - c_3\mathbf{v}_3 - c_4\mathbf{v}_4 = \mathbf{0}$. Row reduction of $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & -\mathbf{v}_3 & -\mathbf{v}_4 & \mathbf{0} \end{bmatrix}$ yields

$$\begin{bmatrix} 5 & 1 & -2 & 0 & 0 \\ 3 & 3 & 1 & 12 & 0 \\ 8 & 4 & -5 & 28 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -10/3 & 0 \\ 0 & 1 & 0 & 26/3 & 0 \\ 0 & 0 & 1 & -4 & 0 \end{bmatrix},$$

so the vector of c_j's must be a multiple of (10/3, –26/3, 4, 1). One simple choice is (10, –26, 12, 3), which gives $\mathbf{w} = 10\mathbf{v}_1 - 26\mathbf{v}_2 = 12\mathbf{v}_3 + 3\mathbf{v}_4 = (24, -48, -24)$. Another choice for **w** is (1, –2, –1).