4.3 SOLUTIONS

Notes: The definition for basis is given initially for subspaces because this emphasizes that the basis elements must be in the subspace. Students often overlook this point when the definition is given for a vector space (see Exercise 25). The subsection on bases for Nul A and Col A is essential for Sections 4.5 and 4.6. The subsection on "Two Views of a Basis" is also fundamental to understanding the interplay between linearly independent sets, spanning sets, and bases. Key exercises in this section are Exercises 21–25, which help to deepen students' understanding of these different subsets of a vector space.

1. Consider the matrix whose columns are the given set of vectors. This 3×3 matrix is in echelon form, and has 3 pivot positions. Thus by the Invertible Matrix Theorem, its columns are linearly independent and span $\mathbb{R}^3$. So the given set of vectors is a basis for $\mathbb{R}^3$.

2. Since the zero vector is a member of the given set of vectors, the set cannot be linearly independent and thus cannot be a basis for $\mathbb{R}^3$. Now consider the matrix whose columns are the given set of vectors. This 3×3 matrix has only 2 pivot positions. Thus by the Invertible Matrix Theorem, its columns do not span $\mathbb{R}^3$.

3. Consider the matrix whose columns are the given set of vectors. The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9/2 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

so the matrix has only two pivot positions. Thus its columns do not form a basis for $\mathbb{R}^3$; the set of vectors is neither linearly independent nor does it span $\mathbb{R}^3$.

4. Consider the matrix whose columns are the given set of vectors. The reduced echelon form of this matrix is

$$\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so the matrix has three pivot positions. Thus its columns form a basis for $\mathbb{R}^3$.

5. Since the zero vector is a member of the given set of vectors, the set cannot be linearly independent and thus cannot be a basis for $\mathbb{R}^3$. Now consider the matrix whose columns are the given set of vectors. The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ -3 & 9 & 0 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so the matrix has a pivot in each row. Thus the given set of vectors spans $\mathbb{R}^3$.

6. Consider the matrix whose columns are the given set of vectors. Since the matrix cannot have a pivot in each row, its columns cannot span $\mathbb{R}^3$; thus the given set of vectors is not a basis for $\mathbb{R}^3$. The reduced echelon form of the matrix is

$$\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

so the matrix has a pivot in each column. Thus the given set of vectors is linearly independent.

7. Consider the matrix whose columns are the given set of vectors. Since the matrix cannot have a pivot in each row, its columns cannot span $\mathbb{R}^3$; thus the given set of vectors is not a basis for $\mathbb{R}^3$. The reduced echelon form of the matrix is

$$\begin{bmatrix} -2 & 6 \\ 3 & -1 \\ 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

so the matrix has a pivot in each column. Thus the given set of vectors is linearly independent.

8. Consider the matrix whose columns are the given set of vectors. Since the matrix cannot have a pivot in each column, the set cannot be linearly independent and thus cannot be a basis for $\mathbb{R}^3$. The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & 2 \\ 3 & -1 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

so the matrix has a pivot in each row. Thus the given set of vectors spans $\mathbb{R}^3$.

9. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables by using the reduced echelon form of A:

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So $x_1 = 3x_3 - 2x_4$, $x_2 = 5x_3 - 4x_4$, with x_3 and x_4 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix},$$

and a basis for Nul A is

$$\left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

10. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables by using the reduced echelon form of A:

$$\begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 7 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}.$$

So $x_1 = 5x_3 - 7x_5$, $x_2 = 4x_3 - 6x_5$, $x_4 = 3x_5$, with x_3 and x_5 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix},$$

and a basis for Nul A is

$$\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

11. Let $A = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$. Then we wish to find a basis for Nul A. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables: $x = -2y - z$ with y and z free. So

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

and a basis for Nul A is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

12. We want to find a basis for the set of vectors in $\mathbb{R}^2$ in the line $5x - y = 0$. Let $A = \begin{bmatrix} 5 & -1 \end{bmatrix}$. Then we wish to find a basis for Nul A. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables: $y = 5x$ with x free. So

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 5 \end{bmatrix},$$

and a basis for Nul A is

$$\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}.$$

13. Since B is a row echelon form of A, we see that the first and second columns of A are its pivot columns. Thus a basis for Col A is

$$\left\{\begin{bmatrix}-2\\2\\-3\end{bmatrix},\begin{bmatrix}4\\-6\\8\end{bmatrix}\right\}.$$

To find a basis for Nul A, we find the general solution of $A\mathbf{x}=\mathbf{0}$ in terms of the free variables: $x_1=-6x_3-5x_4$, $x_2=(-5/2)x_3-(3/2)x_4$, with x_3 and x_4 free. So

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=x_3\begin{bmatrix}-6\\-5/2\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-5\\-3/2\\0\\1\end{bmatrix},$$

and a basis for Nul A is

$$\left\{\begin{bmatrix}-6\\-5/2\\1\\0\end{bmatrix},\begin{bmatrix}-5\\-3/2\\0\\1\end{bmatrix}\right\}.$$

14. Since B is a row echelon form of A, we see that the first, third, and fifth columns of A are its pivot columns. Thus a basis for Col A is

$$\left\{\begin{bmatrix}1\\2\\1\\3\end{bmatrix},\begin{bmatrix}-5\\-5\\0\\-5\end{bmatrix},\begin{bmatrix}-3\\2\\5\\-2\end{bmatrix}\right\}.$$

To find a basis for Nul A, we find the general solution of $A\mathbf{x}=\mathbf{0}$ in terms of the free variables, mentally completing the row reduction of B to get: $x_1=-2x_2-4x_4$, $x_3=(7/5)x_4$, $x_5=0$, with x_2 and x_4 free. So

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-4\\0\\7/5\\1\\0\end{bmatrix},$$

and a basis for Nul A is

$$\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-4\\0\\7/5\\1\\0\end{bmatrix}\right\}.$$

15. This problem is equivalent to finding a basis for Col A, where $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5]$. Since the reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the first, second, and fourth columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}.$$

16. This problem is equivalent to finding a basis for Col A, where $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5]$. Since the reduced echelon form of A is

$$\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the first, second, and third columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix} \right\}.$$

17. **[M]** This problem is equivalent to finding a basis for Col A, where $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5]$. Since the reduced echelon form of A is

$$\begin{bmatrix} 8 & 4 & -1 & 6 & -1 \\ 9 & 5 & -4 & 8 & 4 \\ -3 & 1 & -9 & 4 & 11 \\ -6 & -4 & 6 & -7 & -8 \\ 0 & 4 & -7 & 10 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/2 & 3 \\ 0 & 1 & 0 & 5/2 & -7 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the first, second, and third columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is

$$\left\{ \begin{bmatrix} 8 \\ 9 \\ -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ -4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -9 \\ 6 \\ -7 \end{bmatrix} \right\}.$$

18. **[M]** This problem is equivalent to finding a basis for Col A, where $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5]$. Since the reduced echelon form of A is

$$\begin{bmatrix} -8 & 8 & -8 & 1 & -9 \\ 7 & -7 & 7 & 4 & 3 \\ 6 & -9 & 4 & 9 & -4 \\ 5 & -5 & 5 & 6 & -1 \\ -7 & 7 & -7 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5/3 & 0 & 4/3 \\ 0 & 1 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the first, second, and fourth columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is

$$\left\{ \begin{bmatrix} -8 \\ 7 \\ 6 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 8 \\ -7 \\ -9 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 6 \\ -7 \end{bmatrix} \right\}.$$

19. Since $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$, we see that each of the vectors is a linear combination of the others. Thus the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ all span H. Since we may confirm that none of the three vectors is a multiple of any of the others, the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent and thus each forms a basis for H.

20. Since $\mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$, we see that each of the vectors is a linear combination of the others. Thus the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ all span H. Since we may confirm that none of the three vectors is a multiple of any of the others, the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent and thus each forms a basis for H.

21. **a.** False. The zero vector by itself is linearly dependent. See the paragraph preceding Theorem 4.
b. False. The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ must also be linearly independent. See the definition of a basis.
c. True. See Example 3.
d. False. See the subsection "Two Views of a Basis."
e. False. See the box before Example 9.

22. **a.** False. The subspace spanned by the set must also coincide with H. See the definition of a basis.
b. True. Apply the Spanning Set Theorem to V instead of H. The space V is nonzero because the spanning set uses nonzero vectors.
c. True. See the subsection "Two Views of a Basis."
d. False. See the two paragraphs before Example 8.
e. False. See the warning after Theorem 6.

23. Let $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4]$. Then A is square and its columns span $\mathbb{R}^4$ since $\mathbb{R}^4 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. So its columns are linearly independent by the Invertible Matrix Theorem, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$.

24. Let $A = [\mathbf{v}_1 \quad \ldots \quad \mathbf{v}_n]$. Then A is square and its columns are linearly independent, so its columns span $\mathbb{R}^n$ by the Invertible Matrix Theorem. Thus $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$.

25. In order for the set to be a basis for H, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be a spanning set for H; that is, $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The exercise shows that H is a subset of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. but there are vectors in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ which are not in H ($\mathbf{v}_1$ and $\mathbf{v}_3$, for example). So $H \neq \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis for H.

26. Since $\sin t \cos t = (1/2) \sin 2t$, the set $\{\sin t, \sin 2t\}$ spans the subspace. By inspection we note that this set is linearly independent, so $\{\sin t, \sin 2t\}$ is a basis for the subspace.

27. The set $\{\cos \omega t, \sin \omega t\}$ spans the subspace. By inspection we note that this set is linearly independent, so $\{\cos \omega t, \sin \omega t\}$ is a basis for the subspace.

28. The set $\{e^{-bt}, te^{-bt}\}$ spans the subspace. By inspection we note that this set is linearly independent, so $\{e^{-bt}, te^{-bt}\}$ is a basis for the subspace.

29. Let A be the $n \times k$ matrix $[\mathbf{v}_1 \;\; \ldots \;\; \mathbf{v}_k]$. Since A has fewer columns than rows, there cannot be a pivot position in each row of A. By Theorem 4 in Section 1.4, the columns of A do not span $\mathbb{R}^n$ and thus are not a basis for $\mathbb{R}^n$.

30. Let A be the $n \times k$ matrix $[\mathbf{v}_1 \;\; \ldots \;\; \mathbf{v}_k]$. Since A has fewer rows than columns rows, there cannot be a pivot position in each column of A. By Theorem 8 in Section 1.6, the columns of A are not linearly independent and thus are not a basis for $\mathbb{R}^n$.

31. Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent. Then there exist scalars $c_1, \ldots, c_p$ not all zero with

$$c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}.$$

Since T is linear,

$$T(c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \ldots + c_pT(\mathbf{v}_p)$$

and

$$T(c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p) = T(\mathbf{0}) = \mathbf{0}.$$

Thus

$$c_1T(\mathbf{v}_1) + \ldots + c_pT(\mathbf{v}_p) = \mathbf{0}$$

and since not all of the c_i are zero, $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent.

32. Suppose that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent. Then there exist scalars $c_1, \ldots, c_p$ not all zero with

$$c_1T(\mathbf{v}_1) + \ldots + c_pT(\mathbf{v}_p) = \mathbf{0}.$$

Since T is linear,

$$T(c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \ldots + c_pT(\mathbf{v}_p) = \mathbf{0} = T(\mathbf{0})$$

Since T is one-to-one

$$T(c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p) = T(\mathbf{0})$$

implies that

$$c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}.$$

Since not all of the c_i are zero, $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

33. Neither polynomial is a multiple of the other polynomial. So $\{\mathbf{p}_1, \mathbf{p}_2\}$ is a linearly independent set in $\mathbb{P}_3$. Note: $\{\mathbf{p}_1, \mathbf{p}_2\}$ is also a linearly independent set in $\mathbb{P}_2$ since $\mathbf{p}_1$ and $\mathbf{p}_2$ both happen to be in $\mathbb{P}_2$.

34. By inspection, $\mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_2$, or $\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 = \mathbf{0}$. By the Spanning Set Theorem, $\text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \text{Span}\{\mathbf{p}_1, \mathbf{p}_2\}$. Since neither $\mathbf{p}_1$ nor $\mathbf{p}_2$ is a multiple of the other, they are linearly independent and hence $\{\mathbf{p}_1, \mathbf{p}_2\}$ is a basis for $\text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

35. Let $\{\mathbf{v}_1, \mathbf{v}_3\}$ be any linearly independent set in a vector space V, and let $\mathbf{v}_2$ and $\mathbf{v}_4$ each be linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_3$. For instance, let $\mathbf{v}_2 = 5\mathbf{v}_1$ and $\mathbf{v}_4 = \mathbf{v}_1 + \mathbf{v}_3$. Then $\{\mathbf{v}_1, \mathbf{v}_3\}$ is a basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

36. **[M]** Row reduce the following matrices to identify their pivot columns:

$$[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 3 & -1 & 7 \\ -1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } \{\mathbf{u}_1, \mathbf{u}_2\} \text{ is a basis for } H.$$

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 8 & 9 & 6 \\ -4 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } \{\mathbf{v}_1, \mathbf{v}_2\} \text{ is a basis for } K.$$

$$[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 2 & 1 & 2 & -1 \\ 2 & 2 & 2 & 0 & -2 & 4 \\ 3 & -1 & 7 & 8 & 9 & 6 \\ -1 & 1 & -3 & -4 & -5 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & -4 \\ 0 & 1 & -1 & 0 & -3 & 6 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1\} \text{ is a basis for } H + K.$$

37. **[M]** For example, writing

$$c_1 \cdot t + c_2 \cdot \sin t + c_3 \cos 2t + c_4 \sin t \cos t = 0$$

with $t = 0, .1, .2, .3$ gives the following coefficent matrix A for the homogeneous system $A\mathbf{c} = \mathbf{0}$ (to four decimal places):

$$A = \begin{bmatrix} 0 & \sin 0 & \cos 0 & \sin 0 \cos 0 \\ .1 & \sin .1 & \cos .2 & \sin .1 \cos .1 \\ .2 & \sin .2 & \cos .4 & \sin .2 \cos .2 \\ .3 & \sin .3 & \cos .6 & \sin .3 \cos .3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ .1 & .0998 & .9801 & .0993 \\ .2 & .1987 & .9211 & .1947 \\ .3 & .2955 & .8253 & .2823 \end{bmatrix}.$$

This matrix is invertible, so the system $A\mathbf{c} = \mathbf{0}$ has only the trivial solution and $\{t, \sin t, \cos 2t, \sin t \cos t\}$ is a linearly independent set of functions.

38. **[M]** For example, writing

$$c_1 \cdot 1 + c_2 \cdot \cos t + c_3 \cdot \cos^2 t + c_4 \cdot \cos^3 t + c_5 \cdot \cos^4 t + c_6 \cdot \cos^5 t + c_7 \cdot \cos^6 t = 0$$

with $t = 0, .1, .2, .3, .4, .5, .6$ gives the following coefficent matrix A for the homogeneous system $A\mathbf{c} = \mathbf{0}$ (to four decimal places):

$$A = \begin{bmatrix} 1 & \cos 0 & \cos^2 0 & \cos^3 0 & \cos^4 0 & \cos^5 0 & \cos^6 0 \\ 1 & \cos .1 & \cos^2 .1 & \cos^3 .1 & \cos^4 .1 & \cos^5 .1 & \cos^6 .1 \\ 1 & \cos .2 & \cos^2 .2 & \cos^3 .2 & \cos^4 .2 & \cos^5 .2 & \cos^6 .2 \\ 1 & \cos .3 & \cos^2 .3 & \cos^3 .3 & \cos^4 .3 & \cos^5 .3 & \cos^6 .3 \\ 1 & \cos .4 & \cos^2 .4 & \cos^3 .4 & \cos^4 .4 & \cos^5 .4 & \cos^6 .4 \\ 1 & \cos .5 & \cos^2 .5 & \cos^3 .5 & \cos^4 .5 & \cos^5 .5 & \cos^6 .5 \\ 1 & \cos .6 & \cos^2 .6 & \cos^3 .6 & \cos^4 .6 & \cos^5 .6 & \cos^6 .6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & .9950 & .9900 & .9851 & .9802 & .9753 & .9704 \\ 1 & .9801 & .9605 & .9414 & .9226 & .9042 & .8862 \\ 1 & .9553 & .9127 & .8719 & .8330 & .7958 & .7602 \\ 1 & .9211 & .8484 & .7814 & .7197 & .6629 & .6106 \\ 1 & .8776 & .7702 & .6759 & .5931 & .5205 & .4568 \\ 1 & .8253 & .6812 & .5622 & .4640 & .3830 & .3161 \end{bmatrix}$$

This matrix is invertible, so the system Ac = 0 has only the trivial solution and $\{1, \cos t, \cos^2 t, \cos^3 t, \cos^4 t, \cos^5 t, \cos^6 t\}$ is a linearly independent set of functions.