4.4 SOLUTIONS

Notes: Section 4.7 depends heavily on this section, as does Section 5.4. It is possible to cover the $\mathbb{R}^n$ parts of the two later sections, however, if the first half of Section 4.4 (and perhaps Example 7) is covered. The linearity of the coordinate mapping is used in Section 5.4 to find the matrix of a transformation relative to two bases. The change-of-coordinates matrix appears in Section 5.4, Theorem 8 and Exercise 27. The concept of an isomorphism is needed in the proof of Theorem 17 in Section 4.8. Exercise 25 is used in Section 4.7 to show that the change-of-coordinates matrix is invertible.

1. We calculate that

$$\mathbf{x} = 5\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}.$$

2. We calculate that

$$\mathbf{x} = 8\begin{bmatrix} 4 \\ 5 \end{bmatrix} + (-5)\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

3. We calculate that

$$\mathbf{x} = 3\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}.$$

4. We calculate that

$$\mathbf{x}=(-4)\begin{bmatrix}-1\\2\\0\end{bmatrix}+8\begin{bmatrix}3\\-5\\2\end{bmatrix}+(-7)\begin{bmatrix}4\\-7\\3\end{bmatrix}=\begin{bmatrix}0\\1\\-5\end{bmatrix}.$$

5. The matrix $\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{x}\end{bmatrix}$ row reduces to $\begin{bmatrix}1 & 0 & 8\\0 & 1 & -5\end{bmatrix}$, so $[\mathbf{x}]_B=\begin{bmatrix}8\\-5\end{bmatrix}$.

6. The matrix $\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{x}\end{bmatrix}$ row reduces to $\begin{bmatrix}1 & 0 & -6\\0 & 1 & 2\end{bmatrix}$, so $[\mathbf{x}]_B=\begin{bmatrix}-6\\2\end{bmatrix}$.

7. The matrix $\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{x}\end{bmatrix}$ row reduces to $\begin{bmatrix}1 & 0 & 0 & -1\\0 & 1 & 0 & -1\\0 & 0 & 1 & 3\end{bmatrix}$, so $[\mathbf{x}]_B=\begin{bmatrix}-1\\-1\\3\end{bmatrix}$.

8. The matrix $\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{x}\end{bmatrix}$ row reduces to $\begin{bmatrix}1 & 0 & 0 & -2\\0 & 1 & 0 & 0\\0 & 0 & 1 & 5\end{bmatrix}$, so $[\mathbf{x}]_B=\begin{bmatrix}-2\\0\\5\end{bmatrix}$.

9. The change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^2$ is

$$P_B=\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}=\begin{bmatrix}2 & 1\\-9 & 8\end{bmatrix}.$$

10. The change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^3$ is

$$P_B=\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix}=\begin{bmatrix}3 & 2 & 8\\-1 & 0 & -2\\4 & -5 & 7\end{bmatrix}.$$

11. Since P_B^{-1} converts $\mathbf{x}$ into its B-coordinate vector, we find that

$$[\mathbf{x}]_B=P_B^{-1}\mathbf{x}=\begin{bmatrix}3 & -4\\-5 & 6\end{bmatrix}^{-1}\begin{bmatrix}2\\-6\end{bmatrix}=\begin{bmatrix}-3 & -2\\-5/2 & -3/2\end{bmatrix}\begin{bmatrix}2\\-6\end{bmatrix}=\begin{bmatrix}6\\4\end{bmatrix}.$$

12. Since P_B^{-1} converts $\mathbf{x}$ into its B-coordinate vector, we find that

$$[\mathbf{x}]_B=P_B^{-1}\mathbf{x}=\begin{bmatrix}4 & 6\\5 & 7\end{bmatrix}^{-1}\begin{bmatrix}2\\0\end{bmatrix}=\begin{bmatrix}-7/2 & 3\\5/2 & -2\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}=\begin{bmatrix}-7\\5\end{bmatrix}.$$

13. We must find c_1, c_2, and c_3 such that

$$c_1(1+t^2)+c_2(t+t^2)+c_3(1+2t+t^2)=\mathbf{p}(t)=1+4t+7t^2.$$

Equating the coefficients of the two polynomials produces the system of equations

$$\begin{aligned} c_1 \quad\quad\quad + \quad\; c_3 &= 1\\ c_2 + 2c_3 &= 4\\ c_1 + c_2 + c_3 &= 7 \end{aligned}$$

We row reduce the augmented matrix for the system of equations to find

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \text{ so } [\mathbf{p}]_B = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}.$$

One may also solve this problem using the coordinate vectors of the given polynomials relative to the standard basis $\{1, t, t^2\}$; the same system of linear equations results.

14. We must find c_1, c_2, and c_3 such that

$$c_1(1-t^2)+c_2(t-t^2)+c_3(2-2t+t^2)=\mathbf{p}(t)=3+t-6t^2.$$

Equating the coefficients of the two polynomials produces the system of equations

$$\begin{array}{rcrcrcr} c_1 & + & & & 2c_3 & = & 3 \\ & & c_2 & - & 2c_3 & = & 1 \\ -c_1 & - & c_2 & + & c_3 & = & -6 \end{array}$$

We row reduce the augmented matrix for the system of equations to find

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \text{ so } [\mathbf{p}]_B = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}.$$

One may also solve this problem using the coordinate vectors of the given polynomials relative to the standard basis $\{1, t, t^2\}$; the same system of linear equations results.

15. **a.** True. See the definition of the B-coordinate vector.

b. False. See Equation (4).

c. False. $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$. See Example 5.

16. **a.** True. See Example 2.

b. False. By definition, the coordinate mapping goes in the opposite direction.

c. True. If the plane passes through the origin, as in Example 7, the plane is isomorphic to $\mathbb{R}^2$.

17. We must solve the vector equation $x_1\begin{bmatrix} 1 \\ -3 \end{bmatrix}+x_2\begin{bmatrix} 2 \\ -8 \end{bmatrix}+x_3\begin{bmatrix} -3 \\ 7 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We row reduce the augmented matrix for the system of equations to find

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -3 & -8 & 7 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 5 \\ 0 & 1 & 1 & -2 \end{bmatrix}.$$

Thus we can let $x_1 = 5+5x_3$ and $x_2 = -2-x_3$, where x_3 can be any real number. Letting $x_3 = 0$ and $x_3 = 1$ produces two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of the other vectors: $5\mathbf{v}_1 - 2\mathbf{v}_2$ and $10\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$. There are infintely many correct answers to this problem.

18. For each k, $\mathbf{b}_k = 0\cdot\mathbf{b}_1 + \cdots + 1\cdot\mathbf{b}_k + \cdots + 0\cdot\mathbf{b}_n$, so $[\mathbf{b}_k]_B = (0,\ldots,1,\ldots,0) = \mathbf{e}_k$.

19. The set S spans V because every $\mathbf{x}$ in V has a representation as a (unique) linear combination of elements in S. To show linear independence, suppose that $S = \{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ and that $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ for some scalars $c_1, \ldots, c_n$. The case when $c_1 = \cdots = c_n = 0$ is one possibility. By hypothesis, this is the unique (and thus the only) possible representation of the zero vector as a linear combination of the elements in S. So S is linearly independent and is thus a basis for V.

20. For $\mathbf{w}$ in V there exist scalars k_1, k_2, k_3, and k_4 such that

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 \tag{1}$$

because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ spans V. Because the set is linearly dependent, there exist scalars c_1, c_2, c_3, and c_4 not all zero, such that

$$\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 \tag{2}$$

Adding (1) and (2) gives

$$\mathbf{w} = \mathbf{w} + \mathbf{0} = (k_1 + c_1)\mathbf{v}_1 + (k_2 + c_2)\mathbf{v}_2 + (k_3 + c_3)\mathbf{v}_3 + (k_4 + c_4)\mathbf{v}_4 \tag{3}$$

At least one of the weights in (3) differs from the corresponding weight in (1) because at least one of the c_i is nonzero. So $\mathbf{w}$ is expressed in more than one way as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

21. The matrix of the transformation will be $P_B^{-1} = \begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$.

22. The matrix of the transformation will be $P_B^{-1} = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_n]^{-1}$.

23. Suppose that

$$[\mathbf{u}]_B = [\mathbf{w}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

By definition of coordinate vectors,

$$\mathbf{u} = \mathbf{w} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

Since $\mathbf{u}$ and $\mathbf{w}$ were arbitrary elements of V, the coordinate mapping is one-to-one.

24. Given $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, let $\mathbf{u} = y_1\mathbf{b}_1 + \cdots + y_n\mathbf{b}_n$. Then, by definition, $[\mathbf{u}]_B = \mathbf{y}$. Since $\mathbf{y}$ was arbitrary, the coordinate mapping is onto $\mathbb{R}^n$.

25. Since the coordinate mapping is one-to-one, the following equations have the same solutions $c_1, \ldots, c_p$:

$$c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \mathbf{0} \qquad \text{(the zero vector in } V\text{)} \tag{4}$$

$$\left[c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p\right]_B = [\mathbf{0}]_B \qquad \text{(the zero vector in } \mathbb{R}^n\text{)} \tag{5}$$

Since the coordinate mapping is linear, (5) is equivalent to

$$c_1[\mathbf{u}_1]_B + \cdots + c_p[\mathbf{u}_p]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6}$$

Thus (4) has only the trivial solution if and only if (6) has only the trivial solution. It follows that $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is linearly independent if and only if $\{[\mathbf{u}_1]_B, \ldots, [\mathbf{u}_p]_B\}$ is linearly independent. This result also follows directly from Exercises 31 and 32 in Section 4.3.

26. By definition, $\mathbf{w}$ is a linear combination of $\mathbf{u}_1,\ldots,\mathbf{u}_p$ if and only if there exist scalars $c_1,\ldots,c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p \tag{7}$$

Since the coordinate mapping is linear,

$$[\mathbf{w}]_B = c_1[\mathbf{u}_1]_B + \cdots + c_p[\mathbf{u}_p]_B \tag{8}$$

Conversely, (8) implies (7) because the coordinate mapping is one-to-one. Thus $\mathbf{w}$ is a linear combination of $\mathbf{u}_1,\ldots,\mathbf{u}_p$ if and only if $[\mathbf{w}]_B$ is a linear combination of $[\mathbf{u}]_1,\ldots,[\mathbf{u}]_p$.

Note: Students need to be urged to *write* not just to compute in Exercises 27–34. The language in the *Study Guide* solution of Exercise 31 provides a model for the students. In Exercise 32, students may have difficulty distinguishing between the two isomorphic vector spaces, sometimes giving a vector in $\mathbb{R}^3$ as an answer for part (b).

27. The coordinate mapping produces the coordinate vectors (1, 0, 0, 1), (3, 1, –2, 0), and (0, –1, 3, –1) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix has a pivot in each column, its columns (and thus the given polynomials) are linearly independent.

28. The coordinate mapping produces the coordinate vectors (1, 0, –2, –3), (0, 1, 0, 1), and (1, 3, –2, 0) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ -2 & 0 & -2 \\ -3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are linearly dependent.

29. The coordinate mapping produces the coordinate vectors (1, –2, 1, 0), (–2, 0, 0, 1), and (–8, 12, –6, 1) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & -2 & -8 \\ -2 & 0 & 12 \\ 1 & 0 & -6 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are linearly dependent.

30. The coordinate mapping produces the coordinate vectors (1, –3, 3, –1), (4, –12, 9, 0), and (0, 0, 3, –4) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & 4 & 0 \\ -3 & -12 & 0 \\ 3 & 9 & 3 \\ -1 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are linearly dependent.

31. In each part, place the coordinate vectors of the polynomials into the columns of a matrix and reduce the matrix to echelon form.

a. $\begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 5 & 5 & 0 \\ 5 & -7 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & -4 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Since there is not a pivot in each row, the original four column vectors do not span $\mathbb{R}^3$. By the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the given set of polynomials does not span $\mathbb{P}_2$.

b. $\begin{bmatrix} 0 & 1 & -3 & 2 \\ 5 & -8 & 4 & -3 \\ 1 & -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 2 & -6 & -3 \\ 0 & 0 & 0 & 7/2 \end{bmatrix}$

Since there is a pivot in each row, the original four column vectors span $\mathbb{R}^3$. By the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the given set of polynomials spans $\mathbb{P}_2$.

32. a. Place the coordinate vectors of the polynomials into the columns of a matrix and reduce the matrix to echelon form: $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

The resulting matrix is invertible since it row equivalent to I_3. The original three column vectors form a basis for $\mathbb{R}^3$ by the Invertible Matrix Theorem. By the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the corresponding polynomials form a basis for $\mathbb{P}_2$.

b. Since $[\mathbf{q}]_B = (-3, 1, 2)$, $\mathbf{q} = -3\mathbf{p}_1 + \mathbf{p}_2 + 2\mathbf{p}_3$. One might do the algebra in $\mathbb{P}_2$ or choose to compute $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -8 \end{bmatrix}$. This combination of the columns of the matrix corresponds to the same combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. So $\mathbf{q}(t) = 1 + 3t - 8t^2$.

33. The coordinate mapping produces the coordinate vectors (3, 7, 0, 0), (5, 1, 0, –2), (0, 1, –2, 0) and (1, 16, –6, 2) respectively. To determine whether the set of polynomials is a basis for $\mathbb{P}_3$, we investigate whether the coordinate vectors form a basis for $\mathbb{R}^4$. Writing the vectors as the columns of a matrix and row reducing

$$\begin{bmatrix} 3 & 5 & 0 & 1 \\ 7 & 1 & 1 & 16 \\ 0 & 0 & -2 & -6 \\ 0 & -2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we find that the matrix is not row equivalent to I_4. Thus the coordinate vectors do not form a basis for $\mathbb{R}^4$. By the isomorphism between $\mathbb{R}^4$ and $\mathbb{P}_3$, the given set of polynomials does not form a basis for $\mathbb{P}_3$.

34. The coordinate mapping produces the coordinate vectors (5, –3, 4, 2), (9, 1, 8, –6), (6, –2, 5, 0), and (0, 0, 0, 1) respectively. To determine whether the set of polynomials is a basis for $\mathbb{P}_3$, we investigate whether the coordinate vectors form a basis for $\mathbb{R}^4$. Writing the vectors as the columns of a matrix, and row reducing

$$\begin{bmatrix} 5 & 9 & 6 & 0 \\ -3 & 1 & -2 & 0 \\ 4 & 8 & 5 & 0 \\ 2 & -6 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3/4 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we find that the matrix is not row equivalent to I_4. Thus the coordinate vectors do not form a basis for $\mathbb{R}^4$. By the isomorphism between $\mathbb{R}^4$ and $\mathbb{P}_3$, the given set of polynomials does not form a basis for $\mathbb{P}_3$.

35. To show that $\mathbf{x}$ is in $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, we must show that the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{x}$ has a solution. The augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{x}]$ may be row reduced to show

$$\begin{bmatrix} 11 & 14 & 19 \\ -5 & -8 & -13 \\ 10 & 13 & 18 \\ 7 & 10 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/3 \\ 0 & 1 & 8/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since this system has a solution, $\mathbf{x}$ is in H. The solution allows us to find the B-coordinate vector for $\mathbf{x}$: since $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = (-5/3)\mathbf{v}_1 + (8/3)\mathbf{v}_2$, $[\mathbf{x}]_B = \begin{bmatrix} -5/3 \\ 8/3 \end{bmatrix}$.

36. To show that $\mathbf{x}$ is in $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, we must show that the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{x}$ has a solution. The augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{x}]$ may be row reduced to show

$$\begin{bmatrix} -6 & 8 & -9 & 4 \\ 4 & -3 & 5 & 7 \\ -9 & 7 & -8 & -8 \\ 4 & -3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first three columns show that B is a basis for H. Moreover, since this system has a solution, $\mathbf{x}$ is in H. The solution allows us to find the B-coordinate vector for $\mathbf{x}$: since

$$\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = 3\mathbf{v}_1 + 5\mathbf{v}_2 + 2\mathbf{v}_3,\ [\mathbf{x}]_B = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}.$$

37. We are given that $[\mathbf{x}]_B = \begin{bmatrix} 1/2 \\ 1/4 \\ 1/6 \end{bmatrix}$, where $B = \left\{ \begin{bmatrix} 2.6 \\ -1.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4.8 \end{bmatrix} \right\}$. To find the coordinates of $\mathbf{x}$ relative to the standard basis in $\mathbb{R}^3$, we must find $\mathbf{x}$. We compute that

$$\mathbf{x} = P_B[\mathbf{x}]_B = \begin{bmatrix} 2.6 & 0 & 0 \\ -1.5 & 3 & 0 \\ 0 & 0 & 4.8 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/4 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 0 \\ 0.8 \end{bmatrix}.$$