4.6 SOLUTIONS

Notes: This section puts together most of the ideas from Chapter 4. The Rank Theorem is the main result in this section. Many students have difficulty with the difference in finding bases for the row space and the column space of a matrix. The first process uses the nonzero rows of an echelon form of the matrix. The second process uses the pivots columns of the original matrix, which are usually found through row reduction. Students may also have problems with the varied effects of row operations on the linear dependence relations among the rows and columns of a matrix. Problems of the type found in Exercises 19–26 make excellent test questions. Figure 1 and Example 4 prepare the way for Theorem 3 in Section 6.1; Exercises 27–29 anticipate Example 6 in Section 7.4.

1. The matrix B is in echelon form. There are two pivot columns, so the dimension of Col A is 2. There are two pivot rows, so the dimension of Row A is 2. There are two columns without pivots, so the equation $A\mathbf{x}=\mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2. A basis for Col A is the pivot columns of A:

$$\left\{\begin{bmatrix}1\\-1\\5\end{bmatrix},\begin{bmatrix}-4\\2\\-6\end{bmatrix}\right\}.$$

A basis for Row A is the pivot rows of B: $\{(1,0,-1,5),(0,-2,5,-6)\}$. To find a basis for Nul A row reduce to reduced echelon form:

$$A\sim\begin{bmatrix}1&0&-1&5\\0&1&-5/2&3\end{bmatrix}.$$

The solution to $A\mathbf{x}=\mathbf{0}$ in terms of free variables is $x_1=x_3-5x_4$, $x_2=(5/2)x_3-3x_4$ with x_3 and x_4 free. Thus a basis for Nul A is

$$\left\{\begin{bmatrix}1\\5/2\\1\\0\end{bmatrix},\begin{bmatrix}-5\\-3\\0\\1\end{bmatrix}\right\}.$$

2. The matrix B is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are three pivot rows, so the dimension of Row A is 3. There are two columns without pivots, so the equation $A\mathbf{x}=\mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2. A basis for Col A is the pivot columns of A:

$$\left\{\begin{bmatrix}1\\-2\\-3\\3\end{bmatrix},\begin{bmatrix}4\\-6\\-6\\4\end{bmatrix},\begin{bmatrix}9\\-10\\-3\\0\end{bmatrix}\right\}.$$

A basis for Row A is the pivot rows of B: $\{(1,-3,0,5,-7),(0,0,2,-3,8),(0,0,0,0,5)\}$. To find a basis for Nul A row reduce to reduced echelon form:

$$A\sim\begin{bmatrix}1&-3&0&5&0\\0&0&1&-3/2&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}.$$

The solution to $A\mathbf{x}=\mathbf{0}$ in terms of free variables is $x_1=3x_2-5x_4$, $x_3=(3/2)x_4$, $x_5=0$, with x_2 and x_4 free. Thus a basis for Nul A is

$$\left\{\begin{bmatrix}3\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-5\\0\\3/2\\1\\0\end{bmatrix}\right\}.$$

3. The matrix B is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are three pivot rows, so the dimension of Row A is 3. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2. A basis for Col A is the pivot columns of A:

$$\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\}.$$

A basis for Row A is the pivot rows of B: $\{(2,-3,6,2,5),(0,0,3,-1,1),(0,0,0,1,3)\}$. To find a basis for Nul A row reduce to reduced echelon form:

$$A \sim \begin{bmatrix} 1 & -3/2 & 0 & 0 & -9/2 \\ 0 & 0 & 1 & 0 & 4/3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution to $A\mathbf{x} = \mathbf{0}$ in terms of free variables is $x_1 = (3/2)x_2 + (9/2)x_5$, $x_3 = -(4/3)x_5$, $x_4 = -3x_5$, with x_2 and x_5 free. Thus a basis for Nul A is

$$\left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

4. The matrix B is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are three pivot rows, so the dimension of Row A is 3. There are three columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has three free variables. Thus the dimension of Nul A is 3. A basis for Col A is the pivot columns of A:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{bmatrix} \right\}.$$

A basis for Row A is the pivot rows of B:

$$\{(1,1,-3,7,9,-9),(0,1,-1,3,4,-3),(0,0,0,1,-1,-2)\}.$$

To find a basis for Nul A row reduce to reduced echelon form:

$$A \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution to $A\mathbf{x} = \mathbf{0}$ in terms of free variables is $x_1 = 2x_3 - 9x_5 - 2x_6$, $x_2 = x_3 - 7x_5 - 3x_6$, $x_4 = x_5 + 2x_6$, with x_3, x_5, and x_6 free. Thus a basis for Nul A is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

5. By the Rank Theorem, $\dim \text{Nul } A = 8 - \text{rank } A = 8 - 3 = 5$. Since $\dim \text{Row } A = \text{rank } A, \dim \text{Row } A = 3$. Since $\text{rank } A^T = \dim \text{Col } A^T = \dim \text{Row } A$, $\text{rank} A^T = 3$.

6. By the Rank Theorem, $\dim \text{Nul } A = 3 - \text{rank } A = 3 - 3 = 0$. Since $\dim \text{Row } A = \text{rank } A, \dim \text{Row } A = 3$. Since $\text{rank } A^T = \dim \text{Col } A^T = \dim \text{Row } A$, $\text{rank } A^T = 3$.

7. Yes, Col $A = \mathbb{R}^4$. Since A has four pivot columns, $\dim \text{Col } A = 4$. Thus Col A is a four-dimensional subspace of $\mathbb{R}^4$, and Col $A = \mathbb{R}^4$.
No, Nul $A \neq \mathbb{R}^3$. It is true that $\dim \text{Nul } A = 3$, but Nul A is a subspace of $\mathbb{R}^7$.

8. Since A has four pivot columns, $\text{rank } A = 4$, and $\dim \text{Nul } A = 6 - \text{rank } A = 6 - 4 = 2$.
No. Col $A \neq \mathbb{R}^4$. It is true that $\dim \text{Col } A = \text{rank } A = 4$, but Col A is a subspace of $\mathbb{R}^5$.

9. Since $\dim \text{Nul } A = 4$, $\text{rank } A = 6 - \dim \text{Nul } A = 6 - 4 = 2$. So $\dim \text{Col } A = \text{rank } A = 2$.

10. Since $\dim \text{Nul } A = 5$, $\text{rank } A = 6 - \dim \text{Nul } A = 6 - 5 = 1$. So $\dim \text{Col } A = \text{rank } A = 1$.

11. Since $\dim \text{Nul } A = 2$, $\text{rank } A = 5 - \dim \text{Nul } A = 5 - 2 = 3$. So $\dim \text{Row } A = \dim \text{Col } A = \text{rank } A = 3$.

12. Since $\dim \text{Nul } A = 4$, $\text{rank } A = 6 - \dim \text{Nul } A = 6 - 4 = 2$. So $\dim \text{Row } A = \dim \text{Col } A = \text{rank } A = 2$.

13. The rank of a matrix A equals the number of pivot positions which the matrix has. If A is either a 7×5 matrix or a 5×7 matrix, the largest number of pivot positions that A could have is 5. Thus the largest possible value for rank A is 5.

14. The dimension of the row space of a matrix A is equal to rank A, which equals the number of pivot positions which the matrix has. If A is either a 4×3 matrix or a 3×4 matrix, the largest number of pivot positions that A could have is 3. Thus the largest possible value for dimRow A is 3.

15. Since the rank of A equals the number of pivot positions which the matrix has, and A could have at most 6 pivot positions, $\text{rank } A \leq 6$. Thus $\dim \text{Nul } A = 8 - \text{rank } A \geq 8 - 6 = 2$.

16. Since the rank of A equals the number of pivot positions which the matrix has, and A could have at most 4 pivot positions, $\text{rank } A \leq 4$. Thus $\dim \text{Nul } A = 4 - \text{rank } A \geq 4 - 4 = 0$.

17. **a.** True. The rows of A are identified with the columns of A^T. See the paragraph before Example 1.
b. False. See the warning after Example 2.
c. True. See the Rank Theorem.
d. False. See the Rank Theorem.
e. True. See the Numerical Note before the Practice Problem.

18. **a.** False. Review the warning after Theorem 6 in Section 4.3.

b. False. See the warning after Example 2.

c. True. See the remark in the proof of the Rank Theorem.

d. True. This fact was noted in the paragraph before Example 4. It also follows from the fact that the rows of A^T are the columns of $(A^T)^T = A$.

e. True. See Theorem 13.

19. Yes. Consider the system as $A\mathbf{x} = \mathbf{0}$, where A is a 5×6 matrix. The problem states that $\dim \text{Nul} A = 1$. By the Rank Theorem, $\text{rank}\, A = 6 - \dim\text{Nul}\, A = 5$. Thus $\dim \text{Col}\, A = \text{rank}\, A = 5$, and since Col A is a subspace of $\mathbb{R}^5$, Col $A = \mathbb{R}^5$ So every vector **b** in $\mathbb{R}^5$ is also in Col A, and $A\mathbf{x} = \mathbf{b}$, has a solution for all **b**.

20. No. Consider the system as $A\mathbf{x} = \mathbf{b}$, where A is a 6×8 matrix. The problem states that $\dim\text{Nul}\, A = 2$. By the Rank Theorem, $\text{rank}\, A = 8 - \dim\text{Nul}\, A = 6$. Thus $\dim\text{Col}\, A = \text{rank}\, A = 6$, and since Col A is a subspace of $\mathbb{R}^6$, Col $A = \mathbb{R}^6$ So every vector **b** in $\mathbb{R}^6$ is also in Col A, and $A\mathbf{x} = \mathbf{b}$ has a solution for all **b**. Thus it is impossible to change the entries in **b** to make $A\mathbf{x} = \mathbf{b}$ into an inconsistent system.

21. No. Consider the system as $A\mathbf{x} = \mathbf{b}$, where A is a 9×10 matrix. Since the system has a solution for all **b** in $\mathbb{R}^9$, A must have a pivot in each row, and so $\text{rank} A = 9$. By the Rank Theorem, $\dim\text{Nul} A = 10 - 9 = 1$. Thus it is impossible to find two linearly independent vectors in Nul A.

22. No. Consider the system as $A\mathbf{x} = \mathbf{0}$, where A is a 10×12 matrix. Since A has at most 10 pivot positions, $\text{rank} A \le 10$. By the Rank Theorem, $\dim\text{Nul} A = 12 - \text{rank} A \ge 2$. Thus it is impossible to find a single vector in Nul A which spans Nul A.

23. Yes, six equations are sufficient. Consider the system as $A\mathbf{x} = \mathbf{0}$, where A is a 12×8 matrix. The problem states that $\dim\text{Nul}\, A = 2$. By the Rank Theorem, $\text{rank}\, A = 8 - \dim\text{Nul}\, A = 6$. Thus $\dim\text{Col}\, A = \text{rank}\, A = 6$. So the system $A\mathbf{x} = \mathbf{0}$ is equivalent to the system $B\mathbf{x} = \mathbf{0}$, where B is an echelon form of A with 6 nonzero rows. So the six equations in this system are sufficient to describe the solution set of $A\mathbf{x} = \mathbf{0}$.

24. Yes, No. Consider the system as $A\mathbf{x} = \mathbf{b}$, where A is a 7×6 matrix. Since A has at most 6 pivot positions, $\text{rank}\, A \le 6$. By the Rank Theorem, $\dim\text{Nul}\, A = 6 - \text{rank}\, A \ge 0$. If $\dim\text{Nul}\, A = 0$, then the system $A\mathbf{x} = \mathbf{b}$ will have no free variables. The solution to $A\mathbf{x} = \mathbf{b}$, if it exists, would thus have to be unique. Since $\text{rank}\, A \le 6$, Col A will be a proper subspace of $\mathbb{R}^7$. Thus there exists a **b** in $\mathbb{R}^7$ for which the system $A\mathbf{x} = \mathbf{b}$ is inconsistent, and the system $A\mathbf{x} = \mathbf{b}$ cannot have a unique solution for all **b**.

25. No. Consider the system as $A\mathbf{x} = \mathbf{b}$, where A is a 10×12 matrix. The problem states that $\dim\text{Nul} A = 3$. By the Rank Theorem, $\dim\text{Col}\, A = \text{rank}\, A = 12 - \dim\text{Nul}\, A = 9$. Thus Col A will be a proper subspace of $\mathbb{R}^{10}$ Thus there exists a **b** in $\mathbb{R}^{10}$ for which the system $A\mathbf{x} = \mathbf{b}$ is inconsistent, and the system $A\mathbf{x} = \mathbf{b}$ cannot have a solution for all **b**.

26. Consider the system $A\mathbf{x} = \mathbf{0}$, where A is a $m\times n$ matrix with $m > n$. Since the rank of A is the number of pivot positions that A has and A is assumed to have full rank, $\text{rank}\, A = n$. By the Rank Theorem, $\dim\text{Nul} A = n - \text{rank}\, A = 0$. So $\text{Nul} A = \{\mathbf{0}\}$, and the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens if and only if the columns of A are linearly independent.

27. Since A is an $m \times n$ matrix, Row A is a subspace of $\mathbb{R}^n$, Col A is a subspace of $\mathbb{R}^m$, and Nul A is a subspace of $\mathbb{R}^n$. Likewise since A^T is an $n \times m$ matrix, Row A^T is a subspace of $\mathbb{R}^m$, Col A^T is a subspace of $\mathbb{R}^n$, and $\text{Nul}\,A^T$ is a subspace of $\mathbb{R}^m$. Since $\text{Row}\,A = \text{Col}\,A^T$ and $\text{Col}\,A = \text{Row}\,A^T$, there are four dinstict subspaces in the list: Row A, Col A, Nul A, and $\text{Nul}\,A^T$.

28. a. Since A is an $m \times n$ matrix and dimRow A = rank A,

$$\text{dimRow}\,A + \text{dimNul}\,A = \text{rank}\,A + \text{dimNul}\,A = n.$$

b. Since A^T is an $n \times m$ matrix and $\text{dimCol}\,A = \text{dimRow}\,A = \text{dimCol}\,A^T = \text{rank}\,A^T$,

$$\text{dimCol}\,A + \text{dimNul}\,A^T = \text{rank}\,A^T + \text{dimNul}\,A^T = m.$$

29. Let A be an $m \times n$ matrix. The system $A\mathbf{x} = \mathbf{b}$ will have a solution for all $\mathbf{b}$ in $\mathbb{R}^m$ if and only if A has a pivot position in each row, which happens if and only if dimCol $A = m$. By Exercise 28 b., dimCol $A = m$ if and only if $\text{dimNul}\,A^T = m - m = 0$, or $\text{Nul}\,A^T = \{\mathbf{0}\}$. Finally, $\text{Nul}\,A^T = \{\mathbf{0}\}$ if and only if the equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

30. The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if rank $[A \quad \mathbf{b}]$ = rank A because the two ranks will be equal if and only if $\mathbf{b}$ is not a pivot column of $[A \quad \mathbf{b}]$. The result then follows from Theorem 2 in Section 1.2.

31. Compute that $\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ -3a & -3b & -3c \\ 5a & 5b & 5c \end{bmatrix}$. Each column of $\mathbf{u}\mathbf{v}^T$ is a multiple of $\mathbf{u}$, so $\text{dimCol}\,\mathbf{u}\mathbf{v}^T = 1$, unless $a = b = c = 0$, in which case $\mathbf{u}\mathbf{v}^T$ is the 3×3 zero matrix and $\text{dimCol}\,\mathbf{u}\mathbf{v}^T = 0$. In any case, $\text{rank}\,\mathbf{u}\mathbf{v}^T = \text{dimCol}\,\mathbf{u}\mathbf{v}^T \le 1$

32. Note that the second row of the matrix is twice the first row. Thus if $\mathbf{v} = (1, -3, 4)$, which is the first row of the matrix,

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}.$$

33. Let $A = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$, and assume that rank $A = 1$. Suppose that $\mathbf{u}_1 \neq \mathbf{0}$. Then $\{\mathbf{u}_1\}$ is basis for Col A, since Col A is assumed to be one-dimensional. Thus there are scalars x and y with $\mathbf{u}_2 = x\mathbf{u}_1$ and $\mathbf{u}_3 = y\mathbf{u}_1$, and $A = \mathbf{u}_1\mathbf{v}^T$, where $\mathbf{v} = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$.

If $\mathbf{u}_1 = \mathbf{0}$ but $\mathbf{u}_2 \neq \mathbf{0}$, then similarly $\{\mathbf{u}_2\}$ is basis for Col A, since Col A is assumed to be one-dimensional. Thus there is a scalar x with $\mathbf{u}_3 = x\mathbf{u}_2$, and $A = \mathbf{u}_2\mathbf{v}^T$, where $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix}$.

If $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$ but $\mathbf{u}_3 \neq \mathbf{0}$, then $A = \mathbf{u}_3\mathbf{v}^T$, where $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

34. Let A be an $m \times n$ matrix with of rank $r > 0$, and let U be an echelon form of A. Since A can be reduced to U by row operations, there exist invertible elementary matrices $E_1, \ldots, E_p$ with $(E_p \cdots E_1)A = U$. Thus

$A=(E_p \cdots E_1)^{-1}U$, since the product of invertible matrices is invertible. Let $E=(E_p \cdots E_1)^{-1}$; then $A = EU$. Let the columns of E be denoted by $\mathbf{c}_1,\ldots,\mathbf{c}_m$. Since the rank of A is r, U has r nonzero rows, which can be denoted $\mathbf{d}_1^T,\ldots,\mathbf{d}_r^T$. By the column-row expansion of A (Theorem 10 in Section 2.4):

$$A = EU = \begin{bmatrix}\mathbf{c}_1 & \ldots & \mathbf{c}_m\end{bmatrix}\begin{bmatrix}\mathbf{d}_1^T \\ \vdots \\ \mathbf{d}_r^T \\ \mathbf{0} \\ \vdots \\ \mathbf{0}\end{bmatrix} = \mathbf{c}_1\mathbf{d}_1^T + \ldots + \mathbf{c}_r\mathbf{d}_r^T,$$

which is the sum of r rank 1 matrices.

35. [M]

a. Begin by reducing A to reduced echelon form:

$$A \sim \begin{bmatrix} 1 & 0 & 13/2 & 0 & 5 & 0 & -3 \\ 0 & 1 & 11/2 & 0 & 1/2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -11/2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for Col A is the pivot columns of A, so matrix C contains these columns:

$$C = \begin{bmatrix} 7 & -9 & 5 & -3 \\ -4 & 6 & -2 & -5 \\ 5 & -7 & 5 & 2 \\ -3 & 5 & -1 & -4 \\ 6 & -8 & 4 & 9 \end{bmatrix}.$$

A basis for Row A is the pivot rows of the reduced echelon form of A, so matrix R contains these rows:

$$R = \begin{bmatrix} 1 & 0 & 13/2 & 0 & 5 & 0 & -3 \\ 0 & 1 & 11/2 & 0 & 1/2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -11/2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

To find a basis for Nul A row reduce to reduced echelon form, note that the solution to $A\mathbf{x} = \mathbf{0}$ in terms of free variables is $x_1 = -(13/2)x_3 - 5x_5 + 3x_7$, $x_2 = -(11/2)x_3 - (1/2)x_5 - 2x_7$, $x_4 = (11/2)x_5 - 7x_7$, $x_6 = -x_7$, with x_3, x_5, and x_7 free. Thus matrix N is

$$N = \begin{bmatrix} -13/2 & -5 & 3 \\ -11/2 & -1/2 & -2 \\ 1 & 0 & 0 \\ 0 & 11/2 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

b. The reduced echelon form of A^T is

$$A^T \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -2/11 \\ 0 & 1 & 0 & 0 & -41/11 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 28/11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so the solution to $A^T\mathbf{x} = \mathbf{0}$ in terms of free variables is $x_1 = (2/11)x_5$, $x_2 = (41/11)x_5$, $x_3 = 0$, $x_4 = -(28/11)x_5$, with x_5 free. Thus matrix M is

$$M = \begin{bmatrix} 2/11 \\ 41/11 \\ 0 \\ -28/11 \\ 1 \end{bmatrix}.$$

The matrix $S = \begin{bmatrix} R^T & N \end{bmatrix}$ is 7×7 because the columns of R^T and N are in $\mathbb{R}^7$ and dimRow A + dimNul A = 7. The matrix $T = \begin{bmatrix} C & M \end{bmatrix}$ is 5×5 because the columns of C and M are in $\mathbb{R}^5$ and dimCol A + dimNul $A^T = 5$. Both S and T are invertible because their columns are linearly independent. This fact will be proven in general in Theorem 3 of Section 6.1.

36. **[M]** Answers will vary, but in most cases C will be 6×4, and will be constructed from the first 4 columns of A. In most cases R will be 4×7, N will be 7×3, and M will be 6×2.

37. **[M]** The C and R from Exercise 35 work here, and $A = CR$.

38. **[M]** If A is nonzero, then $A = CR$. Note that $CR = \begin{bmatrix} C\mathbf{r}_1 & C\mathbf{r}_2 & \cdots & C\mathbf{r}_n \end{bmatrix}$, where $\mathbf{r}_1, \ldots, \mathbf{r}_n$ are the columns of R. The columns of R are either pivot columns of R or are not pivot columns of R.

Consider first the pivot columns of R. The i^{th} pivot column of R is $\mathbf{e}_i$, the i^{th} column in the identity matrix, so $C\mathbf{e}_i$ is the i^{th} pivot column of A. Since A and R have pivot columns in the same locations, when C multiplies a pivot column of R, the result is the corresponding pivot column of A in its proper location.

Suppose $\mathbf{r}_j$ is a nonpivot column of R. Then $\mathbf{r}_j$ contains the weights needed to construct the j^{th} column of A from the pivot columns of A, as is discussed in Example 9 of Section 4.3 and in the paragraph preceding that example. Thus $\mathbf{r}_j$ contains the weights needed to construct the j^{th} column of A from the columns of C, and $C\mathbf{r}_j = \mathbf{a}_j$.