5 Eigenvalues and Eigenvectors

5.1 SOLUTIONS

Notes: Exercises 1–6 reinforce the definitions of eigenvalues and eigenvectors. The subsection on eigenvectors and difference equations, along with Exercises 33 and 34, refers to the chapter introductory example and anticipates discussions of dynamical systems in Sections 5.2 and 5.6.

1. The number 2 is an eigenvalue of A if and only if the equation $A\mathbf{x}=2\mathbf{x}$ has a nontrivial solution. This equation is equivalent to $(A-2I)\mathbf{x}=0$. Compute

$$A-2I=\begin{bmatrix}3&2\\3&8\end{bmatrix}-\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}1&2\\3&6\end{bmatrix}$$

The columns of A are obviously linearly dependent, so $(A-2I)\mathbf{x}=0$ has a nontrivial solution, and so 2 is an eigenvalue of A.

2. The number -2 is an eigenvalue of A if and only if the equation $A\mathbf{x}=-2\mathbf{x}$ has a nontrivial solution. This equation is equivalent to $(A+2I)\mathbf{x}=0$. Compute

$$A+2I=\begin{bmatrix}7&3\\3&-1\end{bmatrix}+\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}9&3\\3&1\end{bmatrix}$$

The columns of A are obviously linearly dependent, so $(A+2I)\mathbf{x}=0$ has a nontrivial solution, and so -2 is an eigenvalue of A.

3. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix}-3&1\\-3&8\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix}=\begin{bmatrix}1\\29\end{bmatrix}\neq\lambda\begin{bmatrix}1\\4\end{bmatrix}$. So $\begin{bmatrix}1\\4\end{bmatrix}$ is *not* an eigenvector of A.

4. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix}2&1\\1&4\end{bmatrix}\begin{bmatrix}-1+\sqrt{2}\\1\end{bmatrix}=\begin{bmatrix}-1+2\sqrt{2}\\3+\sqrt{2}\end{bmatrix}$ The second entries of $\mathbf{x}$ and $A\mathbf{x}$ shows that if $A\mathbf{x}$ is a multiple of $\mathbf{x}$, then that multiple must be $3+\sqrt{2}$. Check $3+\sqrt{2}$ times the first entry of $\mathbf{x}$:

$$(3+\sqrt{2})(-1+\sqrt{2})=-3+\left(\sqrt{2}\right)^2+2\sqrt{2}=-1+2\sqrt{2}$$

This matches the first entry of $A\mathbf{x}$, so $\begin{bmatrix}-1+\sqrt{2}\\1\end{bmatrix}$ is an eigenvector of A, and the corresponding eigenvalue is $3+\sqrt{2}$.

5. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ is an eigenvector of A for the eigenvalue 0.

6. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = (-2)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ So $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector of A for the eigenvalue -2.

7. To determine if 4 is an eigenvalue of A, decide if the matrix $A-4I$ is invertible.

$$A-4I = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix}$$

Invertibility can be checked in several ways, but since an eigenvector is needed in the event that one exists, the best strategy is to row reduce the augmented matrix for $(A-4I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 4 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $(A-4I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, so 4 is an eigenvalue. Any nonzero solution of $(A-4I)\mathbf{x} = \mathbf{0}$ is a corresponding eigenvector. The entries in a solution satisfy $x_1 + x_3 = 0$ and $-x_2 - x_3 = 0$, with x_3 free. The general solution is *not* requested, so to save time, simply take any nonzero value for x_3 to produce an eigenvector. If $x_3 = 1$, then $\mathbf{x} = (-1, -1, 1)$.

Note: The answer in the text is $(1, 1, -1)$, written in this form to make the students wonder whether the more common answer given above is also correct. This may initiate a class discussion of what answers are "correct."

8. To determine if 3 is an eigenvalue of A, decide if the matrix $A-3I$ is invertible.

$$A-3I = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Row reducing the augmented matrix $[(A-3I) \quad \mathbf{0}]$ yields:

$$\begin{bmatrix} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $(A-3I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, so 3 is an eigenvalue. Any nonzero solution of $(A-3I)\mathbf{x} = \mathbf{0}$ is a corresponding eigenvector. The entries in a solution satisfy $x_1 - 3x_3 = 0$ and $x_2 - 2x_3 = 0$, with x_3 free. The general solution is *not* requested, so to save time, simply take any nonzero value for x_3 to produce an eigenvector. If $x_3 = 1$, then $\mathbf{x} = (3, 2, 1)$.

9. For $\lambda=1$: $A-1I=\begin{bmatrix}5&0\\2&1\end{bmatrix}-\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}4&0\\2&0\end{bmatrix}$

The augmented matrix for $(A-I)\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix}4&0&0\\2&0&0\end{bmatrix}$. Thus $x_1=0$ and x_2 is free. The general solution of $(A-I)\mathbf{x}=\mathbf{0}$ is $x_2\mathbf{e}_2$, where $\mathbf{e}_2=\begin{bmatrix}0\\1\end{bmatrix}$, and so $\mathbf{e}_2$ is a basis for the eigenspace corresponding to the eigenvalue 1.

For $\lambda=5$: $A-5I=\begin{bmatrix}5&0\\2&1\end{bmatrix}-\begin{bmatrix}5&0\\0&5\end{bmatrix}=\begin{bmatrix}0&0\\2&-4\end{bmatrix}$

The equation $(A-5I)\mathbf{x}=\mathbf{0}$ leads to $2x_1-4x_2=0$, so that $x_1=2x_2$ and x_2 is free. The general solution is $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2x_2\\x_2\end{bmatrix}=x_2\begin{bmatrix}2\\1\end{bmatrix}$. So $\begin{bmatrix}2\\1\end{bmatrix}$ is a basis for the eigenspace.

10. For $\lambda=4$: $A-4I=\begin{bmatrix}10&-9\\4&-2\end{bmatrix}-\begin{bmatrix}4&0\\0&4\end{bmatrix}=\begin{bmatrix}6&-9\\4&-6\end{bmatrix}$.

The augmented matrix for $(A-4I)\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix}6&-9&0\\4&-6&0\end{bmatrix}\sim\begin{bmatrix}1&-9/6&0\\0&0&0\end{bmatrix}$. Thus $x_1=(3/2)x_2$ and x_2 is free. The general solution is $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}(3/2)x_2\\x_2\end{bmatrix}=x_2\begin{bmatrix}3/2\\1\end{bmatrix}$. A basis for the eigenspace corresponding to 4 is $\begin{bmatrix}3/2\\1\end{bmatrix}$. Another choice is $\begin{bmatrix}3\\2\end{bmatrix}$.

11. $A-10I=\begin{bmatrix}4&-2\\-3&9\end{bmatrix}-\begin{bmatrix}10&0\\0&10\end{bmatrix}=\begin{bmatrix}-6&-2\\-3&-1\end{bmatrix}$

The augmented matrix for $(A-10I)\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix}-6&-2&0\\-3&-1&0\end{bmatrix}\sim\begin{bmatrix}1&1/3&0\\0&0&0\end{bmatrix}$. Thus $x_1=(-1/3)x_2$ and x_2 is free. The general solution is $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-(1/3)x_2\\x_2\end{bmatrix}=x_2\begin{bmatrix}-1/3\\1\end{bmatrix}$. A basis for the eigenspace corresponding to 10 is $\begin{bmatrix}-1/3\\1\end{bmatrix}$. Another choice is $\begin{bmatrix}-1\\3\end{bmatrix}$.

12. For $\lambda=1$: $A-I=\begin{bmatrix}7&4\\-3&-1\end{bmatrix}-\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}6&4\\-3&-2\end{bmatrix}$

The augmented matrix for $(A-I)\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix}6&4&0\\-3&-2&0\end{bmatrix}\sim\begin{bmatrix}1&2/3&0\\0&0&0\end{bmatrix}$. Thus $x_1=(-2/3)x_2$ and x_2 is free. A basis for the eigenspace corresponding to 1 is $\begin{bmatrix}-2/3\\1\end{bmatrix}$. Another choice is $\begin{bmatrix}-2\\3\end{bmatrix}$.

For $\lambda=5$: $A-5I=\begin{bmatrix}7&4\\-3&-1\end{bmatrix}-\begin{bmatrix}5&0\\0&5\end{bmatrix}=\begin{bmatrix}2&4\\-3&-6\end{bmatrix}$.

The augmented matrix for $(A-5I)\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = 2x_2$ and x_2 is free.

The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. A basis for the eigenspace is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

13. For $\lambda = 1$:

$$A - 1I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

The equations for $(A-I)\mathbf{x}=\mathbf{0}$ are easy to solve: $\begin{Bmatrix} 3x_1 + x_3 = 0 \\ -2x_1 \qquad = 0 \end{Bmatrix}$

Row operations hardly seem necessary. Obviously x_1 is zero, and hence x_3 is also zero. There are three-variables, so x_2 is free. The general solution of $(A-I)\mathbf{x}=\mathbf{0}$ is $x_2\mathbf{e}_2$, where $\mathbf{e}_2 = (0,1,0)$, and so $\mathbf{e}_2$ provides a basis for the eigenspace.

For $\lambda = 2$:

$$A - 2I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$[(A-2I)\ \ \mathbf{0}] = \begin{bmatrix} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 1/2 & 0 \\ 0 & \textcircled{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = -(1/2)x_3, x_2 = x_3$, with x_3 free. The general solution of $(A-2I)\mathbf{x}=\mathbf{0}$ is $x_3 \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$. A nice basis vector for the eigenspace is $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

For $\lambda = 3$:

$$A - 3I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$$

$$[(A-3I)\ \ \mathbf{0}] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 1 & 0 \\ 0 & \textcircled{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = -x_3, x_2 = x_3$, with x_3 free. A basis vector for the eigenspace is $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

14. For $\lambda = -2$: $A-(-2I) = A+2I = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{bmatrix}$.

The augmented matrix for $[A-(-2)I]\mathbf{x} = \mathbf{0}$, or $(A+2I)\mathbf{x} = \mathbf{0}$, is

$$[(A+2I) \quad \mathbf{0}] = \begin{bmatrix} 3 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & -13 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & -13 & 13/3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $x_1 = (1/3)x_3, x_2 = (1/3)x_3$, with x_3 free. The general solution of $(A+2I)\mathbf{x} = \mathbf{0}$ is $x_3 \begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to -2 is $\begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}$; another is $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

15. For $\lambda = 3$: $[(A-3I) \quad \mathbf{0}] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 2 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 + 2x_2 + 3x_3 = 0$, with x_2 and x_3 free. The general solution of $(A-3I)\mathbf{x} = \mathbf{0}$, is

$$\mathbf{x} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}. \text{ Basis for the eigenspace: } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Note: For simplicity, the text answer omits the set brackets. I permit my students to list a basis without the set brackets. Some instructors may prefer to include brackets.

16. For $\lambda = 4$: $A-4I = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$[(A-4I) \quad \mathbf{0}] = \begin{bmatrix} -1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. So $x_1 = 2x_3, x_2 = 3x_3$, with x_3 and x_4 free variables. The general solution of $(A-4I)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 3x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \text{ Basis for the eigenspace: } \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Note: I urge my students always to include the extra column of zeros when solving a homogeneous system. Exercise 16 provides a situation in which *failing* to add the column is likely to create problems for a student, because the matrix $A-4I$ itself has a column of zeros.

17. The eigenvalues of $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$ are 0, 2, and -1, on the main diagonal, by Theorem 1.

18. The eigenvalues of $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ are 4, 0, and -3, on the main diagonal, by Theorem 1.

19. The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ is not invertible because its columns are linearly dependent. So the number 0 is an eigenvalue of the matrix. See the discussion following Example 5.

20. The matrix $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$ is not invertible because its columns are linearly dependent. So the number 0 is an eigenvalue of A. Eigenvectors for the eigenvalue 0 are solutions of $A\mathbf{x} = \mathbf{0}$ and therefore have entries that produce a linear dependence relation among the columns of A. Any nonzero vector (in $\mathbf{R}^3$) whose entries sum to 0 will work. Find any two such vectors that are not multiples; for instance, $(1, 1, -2)$ and $(1, -1, 0)$.

21. **a.** False. The equation $A\mathbf{x} = \lambda\mathbf{x}$ must have a *nontrivial* solution.
- **b.** True. See the paragraph after Example 5.
- **c.** True. See the discussion of equation (3).
- **d.** True. See Example 2 and the paragraph preceding it. Also, see the Numerical Note.
- **e.** False. See the warning after Example 3.

22. **a.** False. The vector $\mathbf{x}$ in $A\mathbf{x} = \lambda\mathbf{x}$ must be *nonzero*.
- **b.** False. See Example 4 for a two-dimensional eigenspace, which contains two linearly independent eigenvectors corresponding to the same eigenvalue. The statement given is not at all the same as Theorem 2. In fact, it is the *converse* of Theorem 2 (for the case $r = 2$).
- **c.** True. See the paragraph after Example 1.
- **d.** False. Theorem 1 concerns a *triangular* matrix. See Examples 3 and 4 for counterexamples.
- **e.** True. See the paragraph following Example 3. The eigenspace of A corresponding to λ is the null space of the matrix $A - \lambda I$.

23. If a 2×2 matrix A were to have three distinct eigenvalues, then by Theorem 2 there would correspond three linearly independent eigenvectors (one for each eigenvalue). This is impossible because the vectors all belong to a two-dimensional vector space, in which any set of three vectors is linearly dependent. See Theorem 8 in Section 1.7. In general, if an $n\times n$ matrix has p distinct eigenvalues, then by Theorem 2 there would be a linearly independent set of p eigenvectors (one for each eigenvalue). Since these vectors belong to an n-dimensional vector space, p cannot exceed n.

24. A simple example of a 2×2 matrix with only one distinct eigenvalue is a triangular matrix with the same number on the diagonal. By experimentation, one finds that if such a matrix is actually a diagonal matrix then the eigenspace is two dimensional, and otherwise the eigenspace is only one dimensional.

Examples: $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 4 & 5 \\ 0 & 4 \end{bmatrix}$.

25. If λ is an eigenvalue of A, then there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Since A is invertible, $A^{-1}A\mathbf{x} = A^{-1}(\lambda\mathbf{x})$, and so $\mathbf{x} = \lambda(A^{-1}\mathbf{x})$. Since $\mathbf{x} \neq \mathbf{0}$ (and since A is invertible), λ cannot be zero. Then $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, which shows that λ^{-1} is an eigenvalue of A^{-1}.

Note: The *Study Guide* points out here that the relation between the eigenvalues of A and A^{-1} is important in the so-called *inverse power method* for estimating an eigenvalue of a matrix. See Section 5.8.

26. Suppose that A^2 is the zero matrix. If $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$, then $A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. Since $\mathbf{x}$ is nonzero, λ must be nonzero. Thus each eigenvalue of A is zero.

27. Use the *Hint* in the text to write, for any $\lambda, (A - \lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I$. Since $(A - \lambda I)^T$ is invertible if and only if $A - \lambda I$ is invertible (by Theorem 6(c) in Section 2.2), it follows that $A^T - \lambda I$ is *not* invertible if and only if $A - \lambda I$ is *not* invertible. That is, λ is an eigenvalue of A^T if and only if λ is an eigenvalue of A.

Note: If you discuss Exercise 27, you might ask students on a test to show that A and A^T have the same characteristic polynomial (discussed in Section 5.2). Since det $A = \det A^T$, for any square matrix A,

$$\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - (\lambda I)^T) = \det(A - \lambda I).$$

28. If A is lower triangular, then A^T is upper triangular and has the same diagonal entries as A. Hence, by the part of Theorem 1 already proved in the text, these diagonal entries are eigenvalues of A^T. By Exercise 27, they are also eigenvalues of A.

29. Let $\mathbf{v}$ be the vector in $\mathbf{R}^n$ whose entries are all ones. Then $A\mathbf{v} = s\mathbf{v}$.

30. Suppose the column sums of an $n \times n$ matrix A all equal the same number s. By Exercise 29 applied to A^T in place of A, the number s is an eigenvalue of A^T. By Exercise 27, s is an eigenvalue of A.

31. Suppose T reflects points across (or through) a line that passes through the origin. That line consists of all multiples of some nonzero vector $\mathbf{v}$. The points on this line do not move under the action of A. So $T(\mathbf{v}) = \mathbf{v}$. If A is the standard matrix of T, then $A\mathbf{v} = \mathbf{v}$. Thus $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 1. The eigenspace is Span $\{\mathbf{v}\}$. Another eigenspace is generated by any nonzero vector $\mathbf{u}$ that is perpendicular to the given line. (Perpendicularity in $\mathbf{R}^2$ should be a familiar concept even though orthogonality in $\mathbf{R}^n$ has not been discussed yet.) Each vector $\mathbf{x}$ on the line through $\mathbf{u}$ is transformed into the vector $-\mathbf{x}$. The eigenvalue is -1.

33. (The solution is given in the text.)

a. Replace k by $k+1$ in the definition of $\mathbf{x}_k$, and obtain $\mathbf{x}_{k+1} = c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v}$.

b.
$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v}) \\ &= c_1\lambda^k A\mathbf{u} + c_2\mu^k A\mathbf{v} && \text{by linearity} \\ &= c_1\lambda^k \lambda\mathbf{u} + c_2\mu^k \mu\mathbf{v} && \text{since } \mathbf{u} \text{ and } \mathbf{v} \text{ are eigenvectors} \\ &= \mathbf{x}_{k+1} \end{aligned}$$

34. You could try to write $\mathbf{x}_0$ as linear combination of eigenvectors, $\mathbf{v}_1, \ldots, \mathbf{v}_p$. If $\lambda_1, \ldots, \lambda_p$ are corresponding eigenvalues, and if $\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, then you could *define*

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p$$

In this case, for $k = 0, 1, 2, \ldots,$

$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p) \\ &= c_1\lambda_1^k A\mathbf{v}_1 + \cdots + c_p\lambda_p^k A\mathbf{v}_p \quad \text{Linearity} \\ &= c_1\lambda_1^{k+1}\mathbf{v}_1 + \cdots + c_p\lambda_p^{k+1}\mathbf{v}_p \quad \text{The } \mathbf{v}_i \text{ are eigenvectors.} \\ &= \mathbf{x}_{k+1} \end{aligned}$$

35. Using the figure in the exercise, plot $T(\mathbf{u})$ as $2\mathbf{u}$, because **u** is an eigenvector for the eigenvalue 2 of the standard matrix A. Likewise, plot $T(\mathbf{v})$ as $3\mathbf{v}$, because **v** is an eigenvector for the eigenvalue 3. Since T is linear, the image of **w** is $T(\mathbf{w}) = T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

36. As in Exercise 35, $T(\mathbf{u}) = -\mathbf{u}$ and $T(\mathbf{v}) = 3\mathbf{v}$ because **u** and **v** are eigenvectors for the eigenvalues -1 and 3, respectively, of the standard matrix A. Since T is linear, the image of **w** is $T(\mathbf{w}) = T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

Note: The matrix programs supported by this text all have an eigenvalue command. In some cases, such as MATLAB, the command can be structured so it provides eigenvectors as well as a list of the eigenvalues. At this point in the course, students should *not* use the extra power that produces eigenvectors. Students need to be reminded frequently that eigenvectors of A are null vectors of a translate of A. That is why the instructions for Exercises 35–38 tell students to use the method of Example 4.

It is my experience that nearly all students need manual practice finding eigenvectors by the method of Example 4, at least in this section if not also in Sections 5.2 and 5.3. However, [M] exercises do create a burden if eigenvectors must be found manually. For this reason, the data files for the text include a special command, `nulbasis` for each matrix program (MATLAB, Maple, etc.). The output of `nulbasis (A)` is a matrix whose columns provide a basis for the null space of A, and these columns are identical to the ones a student would find by row reducing the augmented matrix $[A \;\; \mathbf{0}]$. With nulbasis, student answers will be the same (up to multiples) as those in the text. I encourage my students to use technology to speed up all numerical homework here, not just the [**M**] exercises,

37. **[M]** Let A be the given matrix. Use the MATLAB commands `eig` and `nulbasis` (or equivalent commands). The command `ev = eig(A)` computes the three eigenvalues of A and stores them in a vector `ev`. In this exercise, `ev` $= (3, 13, 13)$. The eigenspace for the eigenvalue 3 is the null space of $A - 3I$. Use `nulbasis` to produce a basis for each null space. If the format is set for rational display, the result is

$$\texttt{nulbasis(A - ev(1)*eye(3))} = \begin{bmatrix} 5/9 \\ -2/9 \\ 1 \end{bmatrix}.$$

For simplicity, scale the entries by 9. A basis for the eigenspace for $\lambda = 3$: $\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

For the next eigenvalue, 13, compute `nulbasis(A-ev(2)*eye(3))` $= \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Basis for eigenspace for $\lambda = 13$: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

There is no need to use `ev(3)` because it is the same as `ev(2)`.

38. **[M]** `ev = eig(A)` $= (13, -12, -12, 13)$. For $\lambda = 13$:

`nulbasis (A-ev(1)*eye(4))` $= \begin{bmatrix} -1/2 & 1/3 \\ 0 & -4/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Basis for eigenspace: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix} \right\}$

For $\lambda = -12$: `nulbasis(A-ev(2)*eye(4))` $= \begin{bmatrix} 2/7 & 0 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Basis: $\left\{ \begin{bmatrix} 2 \\ 7 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

39. **[M]** For $\lambda = 5$, basis: $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. For $\lambda = -2$, basis: $\left\{ \begin{bmatrix} -2 \\ 7 \\ -5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -5 \\ 0 \\ 5 \end{bmatrix} \right\}$

40. **[M]** `ev = eig(A)` $= (21.68984106239549, -16.68984106239549, 3, 2, 2)$. The first two eigenvalues are the roots of $\lambda^2 - 5\lambda - 362 = 0$.

Basis for $\lambda = \text{ev}(1)$: $\begin{bmatrix} -0.33333333333333 \\ 2.39082008853296 \\ 0.33333333333333 \\ 0.58333333333333 \\ 1.00000000000000 \end{bmatrix}$, for $\lambda = \text{ev}(2)$: $\begin{bmatrix} -0.33333333333333 \\ -0.80748675519962 \\ 0.33333333333333 \\ 0.58333333333333 \\ 1.00000000000000 \end{bmatrix}$.

For the eigenvalues 3 and 2, the eigenbases are $\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -.5 \\ .5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, respectively.

Note: Since so many eigenvalues in text problems are small integers, it is easy for students to form a habit of entering a value for λ in `nulbasis (A-λI)` based on a *visual examination* of the eigenvalues produced by `eig(A)` when only a few decimal places for λ are displayed. Exercise 40 may help your students discover the dangers of this approach.