5.2 SOLUTIONS

Notes: Exercises 9–14 can be omitted, unless you want your students to have some facility with determinants of 3×3 matrices. In later sections, the text will provide eigenvalues when they are needed for matrices larger than 2×2. If you discussed partitioned matrices in Section 2.4, you might wish to bring in Supplementary Exercises 12–14 in Chapter 5. (Also, see Exercise 14 of Section 2.4.)

Exercises 25 and 27 support the subsection on dynamical systems. The calculations in these exercises and Example 5 prepare for the discussion in Section 5.6 about eigenvector decompositions.

1. $A=\begin{bmatrix}2&7\\7&2\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}2&7\\7&2\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}=\begin{bmatrix}2-\lambda&7\\7&2-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(2-\lambda)^2-7^2=4-4\lambda+\lambda^2-49=\lambda^2-4\lambda-45$$

In factored form, the characteristic equation is $(\lambda-9)(\lambda+5)=0$, so the eigenvalues of A are 9 and -5.

2. $A=\begin{bmatrix}5&3\\3&5\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}5-\lambda&3\\3&5-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(5-\lambda)(5-\lambda)-3\cdot3=\lambda^2-10\lambda+16$$

Since $\lambda^2-10\lambda+16=(\lambda-8)(\lambda-2)$, the eigenvalues of A are 8 and 2.

3. $A=\begin{bmatrix}3&-2\\1&-1\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}3-\lambda&-2\\1&-1-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(3-\lambda)(-1-\lambda)-(-2)(1)=\lambda^2-2\lambda-1$$

Use the quadratic formula to solve the characteristic equation and find the eigenvalues:

$$\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{4+4}}{2}=1\pm\sqrt{2}$$

4. $A=\begin{bmatrix}5&-3\\-4&3\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}5-\lambda&-3\\-4&3-\lambda\end{bmatrix}$. The characteristic polynomial of A is

$$\det(A-\lambda I)=(5-\lambda)(3-\lambda)-(-3)(-4)=\lambda^2-8\lambda+3$$

Use the quadratic formula to solve the characteristic equation and find the eigenvalues:

$$\lambda=\frac{8\pm\sqrt{64-4(3)}}{2}=\frac{8\pm2\sqrt{13}}{2}=4\pm\sqrt{13}$$

5. $A=\begin{bmatrix}2&1\\-1&4\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}2-\lambda&1\\-1&4-\lambda\end{bmatrix}$. The characteristic polynomial of A is

$$\det(A-\lambda I)=(2-\lambda)(4-\lambda)-(1)(-1)=\lambda^2-6\lambda+9=(\lambda-3)^2$$

Thus, A has only one eigenvalue 3, with multiplicity 2.

6. $A=\begin{bmatrix}3&-4\\4&8\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}3-\lambda&-4\\4&8-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(3-\lambda)(8-\lambda)-(-4)(4)=\lambda^2-11\lambda+40$$

Use the quadratic formula to solve det $(A-\lambda I)=0$:

$$\lambda=\frac{-11\pm\sqrt{121-4(40)}}{2}=\frac{-11\pm\sqrt{-39}}{2}$$

These values are complex numbers, not real numbers, so A has no real eigenvalues. There is no nonzero vector $\mathbf{x}$ in $\mathbf{R}^2$ such that $A\mathbf{x}=\lambda\mathbf{x}$, because a real vector $A\mathbf{x}$ cannot equal a complex multiple of $\mathbf{x}$.

7. $A=\begin{bmatrix}5 & 3\\ -4 & 4\end{bmatrix}, A-\lambda I=\begin{bmatrix}5-\lambda & 3\\ -4 & 4-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(5-\lambda)(4-\lambda)-(3)(-4)=\lambda^2-9\lambda+32$$

Use the quadratic formula to solve det $(A-\lambda I)=0$:

$$\lambda=\frac{9\pm\sqrt{81-4(32)}}{2}=\frac{9\pm\sqrt{-47}}{2}$$

These values are complex numbers, not real numbers, so A has no real eigenvalues. There is no nonzero vector $\mathbf{x}$ in $\mathbf{R}^2$ such that $A\mathbf{x}=\lambda\mathbf{x}$, because a real vector $A\mathbf{x}$ cannot equal a complex multiple of $\mathbf{x}$.

8. $A=\begin{bmatrix}7 & -2\\ 2 & 3\end{bmatrix}, A-\lambda I=\begin{bmatrix}7-\lambda & -2\\ 2 & 3-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(7-\lambda)(3-\lambda)-(-2)(2)=\lambda^2-10\lambda+25$$

Since $\lambda^2-10\lambda+25=(\lambda-5)^2$, the only eigenvalue is 5, with multiplicity 2.

9. $\det(A-\lambda I)=\det\begin{bmatrix}1-\lambda & 0 & -1\\ 2 & 3-\lambda & -1\\ 0 & 6 & 0-\lambda\end{bmatrix}$. From the special formula for 3×3 determinants, the characteristic polynomial is

$$\begin{aligned}\det(A-\lambda I)&=(1-\lambda)(3-\lambda)(-\lambda)+0+(-1)(2)(6)-0-(6)(-1)(1-\lambda)-0\\ &=(\lambda^2-4\lambda+3)(-\lambda)-12+6(1-\lambda)\\ &=-\lambda^3+4\lambda^2-3\lambda-12+6-6\lambda\\ &=-\lambda^3+4\lambda^2-9\lambda-6\end{aligned}$$

(This polynomial has one irrational zero and two imaginary zeros.) Another way to evaluate the determinant is to interchange rows 1 and 2 (which reverses the sign of the determinant) and then make one row replacement:

$$\det\begin{bmatrix}1-\lambda & 0 & -1\\ 2 & 3-\lambda & -1\\ 0 & 6 & 0-\lambda\end{bmatrix}=-\det\begin{bmatrix}2 & 3-\lambda & -1\\ 1-\lambda & 0 & -1\\ 0 & 6 & 0-\lambda\end{bmatrix}$$

$$=-\det\begin{bmatrix}2 & 3-\lambda & -1\\ 0 & 0+(.5\lambda-.5)(3-\lambda) & -1+(.5\lambda-.5)(-1)\\ 0 & 6 & 0-\lambda\end{bmatrix}$$

Next, expand by cofactors down the first column. The quantity above equals

$$-2\det\begin{bmatrix}(.5\lambda-.5)(3-\lambda) & -.5-.5\lambda\\ 6 & -\lambda\end{bmatrix}=-2[(.5\lambda-.5)(3-\lambda)(-\lambda)-(-.5-.5\lambda)(6)]$$

$$=(1-\lambda)(3-\lambda)(-\lambda)-(1+\lambda)(6)=(\lambda^2-4\lambda+3)(-\lambda)-6-6\lambda=-\lambda^3+4\lambda^2-9\lambda-6$$

10. $\det(A-\lambda I)=\det\begin{bmatrix}0-\lambda & 3 & 1\\ 3 & 0-\lambda & 2\\ 1 & 2 & 0-\lambda\end{bmatrix}$. From the special formula for 3×3 determinants, the characteristic polynomial is

$$\det(A-\lambda I)=(-\lambda)(-\lambda)(-\lambda)+3\cdot 2\cdot 1+1\cdot 3\cdot 2-1\cdot(-\lambda)\cdot 1-2\cdot 2\cdot(-\lambda)-(-\lambda)\cdot 3\cdot 3$$
$$=-\lambda^3+6+6+\lambda+4\lambda+9\lambda=-\lambda^3+14\lambda+12$$

11. The special arrangements of zeros in A makes a cofactor expansion along the first row highly effective.

$$\det(A-\lambda I)=\det\begin{bmatrix}4-\lambda & 0 & 0\\ 5 & 3-\lambda & 2\\ -2 & 0 & 2-\lambda\end{bmatrix}=(4-\lambda)\det\begin{bmatrix}3-\lambda & 2\\ 0 & 2-\lambda\end{bmatrix}$$
$$=(4-\lambda)(3-\lambda)(2-\lambda)=(4-\lambda)(\lambda^2-5\lambda+6)=-\lambda^3+9\lambda^2-26\lambda+24$$

If only the eigenvalues were required, there would be no need here to write the characteristic polynomial in expanded form.

12. Make a cofactor expansion along the third row:

$$\det(A-\lambda I)=\det\begin{bmatrix}-1-\lambda & 0 & 1\\ -3 & 4-\lambda & 1\\ 0 & 0 & 2-\lambda\end{bmatrix}=(2-\lambda)\cdot\det\begin{bmatrix}-1-\lambda & 0\\ -3 & 4-\lambda\end{bmatrix}$$
$$=(2-\lambda)(-1-\lambda)(4-\lambda)=-\lambda^3+5\lambda^2-2\lambda-8$$

13. Make a cofactor expansion down the third column:

$$\det(A-\lambda I)=\det\begin{bmatrix}6-\lambda & -2 & 0\\ -2 & 9-\lambda & 0\\ 5 & 8 & 3-\lambda\end{bmatrix}=(3-\lambda)\cdot\det\begin{bmatrix}6-\lambda & -2\\ -2 & 9-\lambda\end{bmatrix}$$
$$=(3-\lambda)[(6-\lambda)(9-\lambda)-(-2)(-2)]=(3-\lambda)(\lambda^2-15\lambda+50)$$
$$=-\lambda^3+18\lambda^2-95\lambda+150 \text{ or } (3-\lambda)(\lambda-5)(\lambda-10)$$

14. Make a cofactor expansion along the second row:

$$\det(A-\lambda I)=\det\begin{bmatrix}5-\lambda & -2 & 3\\ 0 & 1-\lambda & 0\\ 6 & 7 & -2-\lambda\end{bmatrix}=(1-\lambda)\cdot\det\begin{bmatrix}5-\lambda & 3\\ 6 & -2-\lambda\end{bmatrix}$$
$$=(1-\lambda)\cdot[(5-\lambda)(-2-\lambda)-3\cdot 6]=(1-\lambda)(\lambda^2-3\lambda-28)$$
$$=-\lambda^3+4\lambda^2+25\lambda-28 \text{ or } (1-\lambda)(\lambda-7)(\lambda+4)$$

15. Use the fact that the determinant of a triangular matrix is the product of the diagonal entries:

$$\det(A-\lambda I)=\det\begin{bmatrix}4-\lambda & -7 & 0 & 2\\ 0 & 3-\lambda & -4 & 6\\ 0 & 0 & 3-\lambda & -8\\ 0 & 0 & 0 & 1-\lambda\end{bmatrix}=(4-\lambda)(3-\lambda)^2(1-\lambda)$$

The eigenvalues are 4, 3, 3, and 1.

16. The determinant of a triangular matrix is the product of its diagonal entries:

$$\det(A-\lambda I)=\det\begin{bmatrix}5-\lambda & 0 & 0 & 0\\ 8 & -4-\lambda & 0 & 0\\ 0 & 7 & 1-\lambda & 0\\ 1 & -5 & 2 & 1-\lambda\end{bmatrix}=(5-\lambda)(-4-\lambda)(1-\lambda)^2$$

The eigenvalues are 5, 1, 1, and –4.

17. The determinant of a triangular matrix is the product of its diagonal entries:

$$\begin{bmatrix}3-\lambda & 0 & 0 & 0 & 0\\ -5 & 1-\lambda & 0 & 0 & 0\\ 3 & 8 & 0-\lambda & 0 & 0\\ 0 & -7 & 2 & 1-\lambda & 0\\ -4 & 1 & 9 & -2 & 3-\lambda\end{bmatrix}=(3-\lambda)^2(1-\lambda)^2(-\lambda)$$

The eigenvalues are 3, 3, 1, 1, and 0.

18. Row reduce the augmented matrix for the equation $(A-5I)\mathbf{x}=\mathbf{0}$:

$$\begin{bmatrix}0 & -2 & 6 & -1 & 0\\ 0 & -2 & h & 0 & 0\\ 0 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & -4 & 0\end{bmatrix}\sim\begin{bmatrix}0 & -2 & 6 & -1 & 0\\ 0 & 0 & h-6 & 1 & 0\\ 0 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 4 & 0\end{bmatrix}\sim\begin{bmatrix}0 & 1 & -3 & 0 & 0\\ 0 & 0 & h-6 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

For a two-dimensional eigenspace, the system above needs two free variables. This happens if and only if $h=6$.

19. Since the equation $\det(A-\lambda I)=(\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$ holds for all λ, set $\lambda=0$ and conclude that $\det A=\lambda_1\lambda_2\cdots\lambda_n$.

20. $\det(A^T-\lambda I)=\det(A^T-\lambda I^T)$

$=\det(A-\lambda I)^T$ Transpose property

$=\det(A-\lambda I)$ Theorem 3(c)

21. **a.** False. See Example 1.
b. False. See Theorem 3.
c. True. See Theorem 3.
d. False. See the solution of Example 4.

22. **a.** False. See the paragraph before Theorem 3.
b. False. See Theorem 3.
c. True. See the paragraph before Example 4.
d. False. See the warning after Theorem 4.

23. If $A=QR$, with Q invertible, and if $A_1=RQ$, then write $A_1=Q^{-1}QRQ=Q^{-1}AQ$, which shows that A_1 is similar to A.

24. First, observe that if P is invertible, then Theorem 3(b) shows that

$$1 = \det I = \det(PP^{-1}) = (\det P)(\det P^{-1})$$

Use Theorem 3(b) again when $A = PBP^{-1}$,

$$\det A = \det(PBP^{-1}) = (\det P)(\det B)(\det P^{-1}) = (\det B)(\det P)(\det P^{-1}) = \det B$$

25. Example 5 of Section 4.9 showed that $A\mathbf{v}_1 = \mathbf{v}_1$, which means that $\mathbf{v}_1$ is an eigenvector of A corresponding to the eigenvalue 1.

a. Since A is a 2×2 matrix, the eigenvalues are easy to find, and factoring the characteristic polynomial is easy when one of the two factors is known.

$$\det\begin{bmatrix} .6-\lambda & .3 \\ .4 & .7-\lambda \end{bmatrix} = (.6-\lambda)(.7-\lambda)-(.3)(.4) = \lambda^2 - 1.3\lambda + .3 = (\lambda-1)(\lambda-.3)$$

The eigenvalues are 1 and .3. For the eigenvalue .3, solve $(A-.3I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} .6-.3 & .3 & 0 \\ .4 & .7-.3 & 0 \end{bmatrix} = \begin{bmatrix} .3 & .3 & 0 \\ .4 & .4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here $x_1 - x_2 = 0$, with x_2 free. The general solution is not needed. Set $x_2 = 1$ to find an eigenvector $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. A suitable basis for $\mathbf{R}^2$ is $\{\mathbf{v}_1, \mathbf{v}_2\}$.

b. Write $\mathbf{x}_0 = \mathbf{v}_1 + c\mathbf{v}_2$: $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. By inspection, c is $-1/14$. (The value of c depends on how $\mathbf{v}_2$ is scaled.)

c. For $k = 1, 2, \dots$, define $\mathbf{x}_k = A^k\mathbf{x}_0$. Then $\mathbf{x}_1 = A(\mathbf{v}_1 + c\mathbf{v}_2) = A\mathbf{v}_1 + cA\mathbf{v}_2 = \mathbf{v}_1 + c(.3)\mathbf{v}_2$, because $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors. Again

$$\mathbf{x}_2 = A\mathbf{x}_1 = A(\mathbf{v}_1 + c(.3)\mathbf{v}_2) = A\mathbf{v}_1 + c(.3)A\mathbf{v}_2 = \mathbf{v}_1 + c(.3)(.3)\mathbf{v}_2.$$

Continuing, the general pattern is $\mathbf{x}_k = \mathbf{v}_1 + c(.3)^k\mathbf{v}_2$. As k increases, the second term tends to $\mathbf{0}$ and so $\mathbf{x}_k$ tends to $\mathbf{v}_1$.

26. If $a \neq 0$, then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & b \\ 0 & d - ca^{-1}b \end{bmatrix} = U$, and $\det A = (a)(d - ca^{-1}b) = ad - bc$. If $a = 0$, then $A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} = U$ (with one interchange), so $\det A = (-1)^1(cb) = 0 - bc = ad - bc$.

27. a. $A\mathbf{v}_1 = \mathbf{v}_1$, $A\mathbf{v}_2 = .5\mathbf{v}_2$, $A\mathbf{v}_3 = .2\mathbf{v}_3$.

b. The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent because the eigenvectors correspond to different eigenvalues (Theorem 2). Since there are three vectors in the set, the set is a basis for $\mathbb{R}^3$. So there exist unique constants such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, and $\mathbf{w}^T\mathbf{x}_0 = c_1\mathbf{w}^T\mathbf{v}_1 + c_2\mathbf{w}^T\mathbf{v}_2 + c_3\mathbf{w}^T\mathbf{v}_3$. Since $\mathbf{x}_0$ and $\mathbf{v}_1$ are probability vectors and since the entries in $\mathbf{v}_2$ and $\mathbf{v}_3$ sum to 0, the above equation shows that $c_1 = 1$.

c. By (b), $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Using (a),

$$\mathbf{x}_k = A^k\mathbf{x}_0 = c_1A^k\mathbf{v}_1 + c_2A^k\mathbf{v}_2 + c_3A^k\mathbf{v}_3 = \mathbf{v}_1 + c_2(.5)^k\mathbf{v}_2 + c_3(.2)^k\mathbf{v}_3 \to \mathbf{v}_1 \text{ as } k \to \infty$$

28. [M]

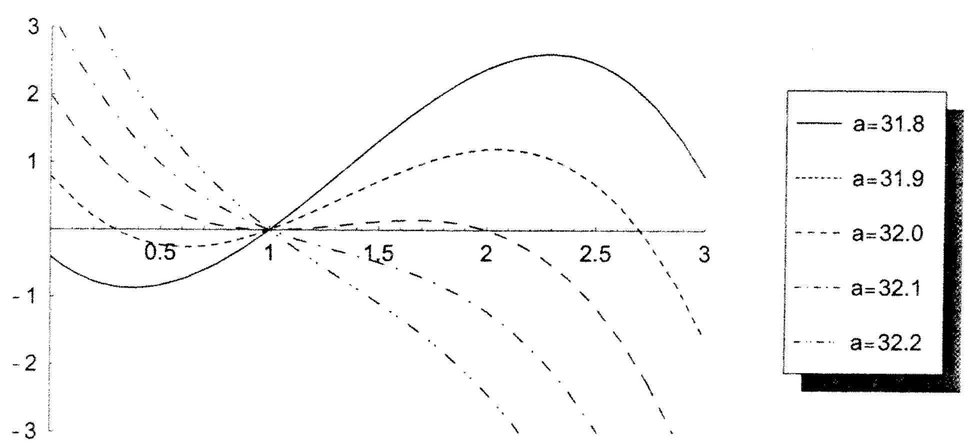
Answers will vary, but should show that the eigenvectors of A are not the same as the eigenvectors of A^T, unless, of course, $A^T = A$.

29. [M] Answers will vary. The product of the eigenvalues of A should equal det A.

30. [M] The characteristic polynomials and the eigenvalues for the various values of a are given in the following table:

a	Characteristic Polynomial	Eigenvalues
31.8	$-.4-2.6t+4t^2-t^3$	$3.1279, 1, -.1279$
31.9	$.8-3.8t+4t^2-t^3$	$2.7042, 1, .2958$
32.0	$2-5t+4t^2-t^3$	$2, 1, 1$
32.1	$3.2-6.2t+4t^2-t^3$	$1.5\pm.9747i, 1$
32.2	$4.4-7.4t+4t^2-t^3$	$1.5\pm1.4663i, 1$

The graphs of the characteristic polynomials are:



Notes: An appendix in Section 5.3 of the *Study Guide* gives an example of factoring a cubic polynomial with integer coefficients, in case you want your students to find integer eigenvalues of simple 3×3 or perhaps 4×4 matrices.

The MATLAB box for Section 5.3 introduces the command `poly (A)`, which lists the coefficients of the characteristic polynomial of the matrix A, and it gives MATLAB code that will produce a graph of the characteristic polynomial. (This is needed for Exercise 30.) The Maple and Mathematica appendices have corresponding information. The appendices for the TI and HP calculators contain only the commands that list the coefficients of the characteristic polynomial.