5.3 SOLUTIONS

1. $P=\begin{bmatrix}5&7\\2&3\end{bmatrix}, D=\begin{bmatrix}2&0\\0&1\end{bmatrix}, A=PDP^{-1}$, and $A^4=PD^4P^{-1}$. We compute $P^{-1}=\begin{bmatrix}3&-7\\-2&5\end{bmatrix}, D^4=\begin{bmatrix}16&0\\0&1\end{bmatrix}$, and $A^4=\begin{bmatrix}5&7\\2&3\end{bmatrix}\begin{bmatrix}16&0\\0&1\end{bmatrix}\begin{bmatrix}3&-7\\-2&5\end{bmatrix}=\begin{bmatrix}226&-525\\90&-209\end{bmatrix}$

2. $P=\begin{bmatrix}2&-3\\-3&5\end{bmatrix}, D=\begin{bmatrix}1&0\\0&1/2\end{bmatrix}, A=PDP^{-1}$, and $A^4=PD^4P^{-1}$. We compute

$P^{-1}=\begin{bmatrix}5&3\\3&2\end{bmatrix}, D^4=\begin{bmatrix}1&0\\0&1/16\end{bmatrix}$, and $A^4=\begin{bmatrix}2&-3\\-3&5\end{bmatrix}\begin{bmatrix}1&0\\0&1/16\end{bmatrix}\begin{bmatrix}5&3\\3&2\end{bmatrix}=\frac{1}{16}\begin{bmatrix}151&90\\-225&-134\end{bmatrix}$

3. $A^k=PD^kP^{-1}=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}a^k&0\\0&b^k\end{bmatrix}\begin{bmatrix}1&0\\-3&1\end{bmatrix}=\begin{bmatrix}a^k&0\\3a^k-3b^k&b^k\end{bmatrix}.$

4. $A^k=PD^kP^{-1}=\begin{bmatrix}3&4\\1&1\end{bmatrix}\begin{bmatrix}2^k&0\\0&1^k\end{bmatrix}\begin{bmatrix}-1&4\\1&-3\end{bmatrix}=\begin{bmatrix}4-3\cdot 2^k&12\cdot 2^k-12\\1-2^k&4\cdot 2^k-3\end{bmatrix}.$

5. By the Diagonalization Theorem, eigenvectors form the columns of the left factor, and they correspond respectively to the eigenvalues on the diagonal of the middle factor.

$$\lambda=5:\begin{bmatrix}1\\1\\1\end{bmatrix};\lambda=1:\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}2\\-1\\0\end{bmatrix}$$

6. As in Exercise 5, inspection of the factorization gives:

$$\lambda=4:\begin{bmatrix}-1\\2\\0\end{bmatrix};\lambda=5:\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}$$

7. Since A is triangular, its eigenvalues are obviously ± 1.

<u>For $\lambda=1$</u>: $A-1I=\begin{bmatrix}0&0\\6&-2\end{bmatrix}$. The equation $(A-1I)\mathbf{x}=\mathbf{0}$ amounts to $6x_1-2x_2=0$, so $x_1=(1/3)x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}1/3\\1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\3\end{bmatrix}$.

<u>For $\lambda=-1$</u>: $A+1I=\begin{bmatrix}2&0\\6&0\end{bmatrix}$. The equation $(A+1I)\mathbf{x}=\mathbf{0}$ amounts to $2x_1=0$, so $x_1=0$ with x_2 free. The general solution is $x_2\begin{bmatrix}0\\1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_2=\begin{bmatrix}0\\1\end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ construct $P=\begin{bmatrix}\mathbf{v}_1&\mathbf{v}_2\end{bmatrix}=\begin{bmatrix}1&0\\3&1\end{bmatrix}$. Then set $D=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

8. Since A is triangular, its only eigenvalue is obviously 5.

For $\lambda = 5$: $A-5I=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The equation $(A-5I)\mathbf{x}=\mathbf{0}$ amounts to $x_2 = 0$, so $x_2 = 0$ with x_1 free. The general solution is $x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since we cannot generate an eigenvector basis for $\mathbb{R}^2$, A is not diagonalizable.

9. To find the eigenvalues of A, compute its characteristic polynomial:

$$\det(A-\lambda I)=\det\begin{bmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{bmatrix}=(3-\lambda)(5-\lambda)-(-1)(1)=\lambda^2-8\lambda+16=(\lambda-4)^2$$

Thus the only eigenvalue of A is 4.

For $\lambda = 4$: $A-4I=\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. The equation $(A-4I)\mathbf{x}=\mathbf{0}$ amounts to $x_1+x_2=0$, so $x_1=-x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Since we cannot generate an eigenvector basis for $\mathbb{R}^2$, A is not diagonalizable.

10. To find the eigenvalues of A, compute its characteristic polynomial:

$$\det(A-\lambda I)=\det\begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix}=(2-\lambda)(1-\lambda)-(3)(4)=\lambda^2-3\lambda-10=(\lambda-5)(\lambda+2)$$

Thus the eigenvalues of A are 5 and -2.

For $\lambda = 5$: $A-5I=\begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}$. The equation $(A-5I)\mathbf{x}=\mathbf{0}$ amounts to $x_1-x_2=0$, so $x_1=x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = -2$: $A+2I=\begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}$. The equation $(A+1I)\mathbf{x}=\mathbf{0}$ amounts to $4x_1+3x_2=0$, so $x_1=(-3/4)x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_2=\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ construct $P=\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}=\begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$. Then set $D=\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

11. The eigenvalues of A are given to be 1, 2, and 3.

For $\lambda = 3$: $A-3I=\begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-3I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3\begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

<u>For $\lambda = 2$</u>: $A-2I=\begin{bmatrix}-3&4&-2\\-3&2&0\\-3&1&1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-2I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&-2/3&0\\0&1&-1&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}2/3\\1\\1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_2=\begin{bmatrix}2\\3\\3\end{bmatrix}$.

<u>For $\lambda = 1$</u>: $A-I=\begin{bmatrix}-2&4&-2\\-3&3&0\\-3&1&2\end{bmatrix}$, and row reducing $\begin{bmatrix}A-1I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_3=\begin{bmatrix}1\\1\\1\end{bmatrix}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}1&2&1\\3&3&1\\4&3&1\end{bmatrix}$. Then set $D=\begin{bmatrix}3&0&0\\0&2&0\\0&0&1\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

12. The eigenvalues of A are given to be 2 and 8.

<u>For $\lambda = 8$</u>: $A-8I=\begin{bmatrix}-4&2&2\\2&-4&2\\2&2&-4\end{bmatrix}$, and row reducing $\begin{bmatrix}A-8I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$.

<u>For $\lambda = 2$</u>: $A-2I=\begin{bmatrix}2&2&2\\2&2&2\\2&2&2\end{bmatrix}$, and row reducing $\begin{bmatrix}A-2I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&1&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}-1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}1&-1&-1\\1&1&0\\1&0&1\end{bmatrix}$. Then set $D=\begin{bmatrix}8&0&0\\0&2&0\\0&0&2\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

13. The eigenvalues of A are given to be 5 and 1.

For $\lambda = 5$: $A-5I=\begin{bmatrix}-3 & 2 & -1\\ 1 & -2 & -1\\ -1 & -2 & -3\end{bmatrix}$, and row reducing $\begin{bmatrix}A-5I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$.

For $\lambda = 1$: $A-1I=\begin{bmatrix}1 & 2 & -1\\ 1 & 2 & -1\\ -1 & -2 & 1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 2 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}\right\}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}-1 & -2 & 1\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$. Then set $D=\begin{bmatrix}5 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

14. The eigenvalues of A are given to be 5 and 4.

For $\lambda = 5$: $A-5I=\begin{bmatrix}-1 & 0 & -2\\ 2 & 0 & 4\\ 0 & 0 & 0\end{bmatrix}$, and row reducing $\begin{bmatrix}A-5I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & 2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}-2\\ 0\\ 1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix}-2\\ 0\\ 1\end{bmatrix},\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\right\}$.

For $\lambda = 4$: $A-4I=\begin{bmatrix}0 & 0 & -2\\ 2 & 1 & 4\\ 0 & 0 & 1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-4I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 1/2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}-1/2\\ 1\\ 0\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_3=\begin{bmatrix}-1\\ 2\\ 0\end{bmatrix}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}-2 & 0 & -1\\ 0 & 1 & 2\\ 1 & 0 & 0\end{bmatrix}$. Then set $D=\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 4\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

15. The eigenvalues of A are given to be 3 and 1.

For $\lambda = 3$: $A-3I=\begin{bmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-3I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}\right\}$

For $\lambda = 1$: $A-I=\begin{bmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_3=\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}=\begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Then set $D=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

16. The eigenvalues of A are given to be 2 and 1.

For $\lambda = 2$: $A-2I=\begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-2I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$.

For $\lambda = 1$: $A-I=\begin{bmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_3=\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}=\begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Then set $D=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

17. Since A is triangular, its eigenvalues are obviously 4 and 5.

For $\lambda = 4$: $A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and row reducing $[A - 4I \quad \mathbf{0}]$ yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Since $\lambda = 5$ must have only a one-dimensional eigenspace, we can find at most 2 linearly independent eigenvectors for A, so A is not diagonalizable.

18. An eigenvalue of A is given to be 5; an eigenvector $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is also given. To find the eigenvalue corresponding to $\mathbf{v}_1$, compute $A\mathbf{v}_1 = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 6 \end{bmatrix} = -3\mathbf{v}_1$. Thus the eigenvalue in question is -3.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} -12 & -16 & 4 \\ 6 & 8 & -2 \\ 12 & 16 & -4 \end{bmatrix}$, and row reducing $[A - 5I \quad \mathbf{0}]$ yields $\begin{bmatrix} 1 & 4/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution is $x_2 \begin{bmatrix} -4/3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$, and a nice basis for the eigenspace is

$\{\mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{bmatrix} -2 & -4 & 1 \\ 1 & 3 & 0 \\ 2 & 0 & 3 \end{bmatrix}$. Then set $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively. Note that this answer differs from the text. There, $P = [\mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_1]$ and the entries in D are rearranged to match the new order of the eigenvectors. According to the Diagonalization Theorem, both answers are correct.

19. Since A is triangular, its eigenvalues are obviously 2, 3, and 5.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and row reducing $[A - 2I \quad \mathbf{0}]$ yields $\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, and a nice basis for the eigenspace is $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 3$: $A-3I=\begin{bmatrix}2 & -3 & 0 & 9\\0 & 0 & 1 & -2\\0 & 0 & -1 & 0\\0 & 0 & 0 & -1\end{bmatrix}$, and row reducing $[A-3I\ \ \mathbf{0}]$ yields $\begin{bmatrix}1 & -3/2 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$.

The general solution is $x_2\begin{bmatrix}3/2\\1\\0\\0\end{bmatrix}$, and a nice basis for the eigenspace is $\mathbf{v}_3=\begin{bmatrix}3\\2\\0\\0\end{bmatrix}$.

For $\lambda = 5$: $A-5I=\begin{bmatrix}0 & -3 & 0 & 9\\0 & -2 & 1 & -2\\0 & 0 & -3 & 0\\0 & 0 & 0 & -3\end{bmatrix}$, and row reducing $[A-5I\ \ \mathbf{0}]$ yields $\begin{bmatrix}0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_4=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$.

From $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ and $\mathbf{v}_4$ construct $P=[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]=\begin{bmatrix}-1 & -1 & 3 & 1\\-1 & 2 & 2 & 0\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}$. Then set $D=\begin{bmatrix}2 & 0 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & 3 & 0\\0 & 0 & 0 & 5\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively. Note that this answer differs from the text. There, $P=[\mathbf{v}_4\ \mathbf{v}_3\ \mathbf{v}_1\ \mathbf{v}_2]$ and the entries in D are rearranged to match the new order of the eigenvectors. According to the Diagonalization Theorem, both answers are correct.

20. Since A is triangular, its eigenvalues are obviously 4 and 2.

For $\lambda = 4$: $A-4I=\begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & -2 & 0\\1 & 0 & 0 & -2\end{bmatrix}$, and row reducing $[A-4I\ \ \mathbf{0}]$ yields $\begin{bmatrix}1 & 0 & 0 & -2 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}2\\0\\0\\1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}2\\0\\0\\1\end{bmatrix}\right\}$.

For $\lambda = 2$: $A-2I=\begin{bmatrix}2 & 0 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & 0 & 0\\1 & 0 & 0 & 0\end{bmatrix}$, and row reducing $[A-2I\ \ \mathbf{0}]$ yields $\begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_3,\mathbf{v}_4\}=\left\{\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

From $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

21. a. False. The symbol D does not automatically denote a diagonal matrix.
    b. True. See the remark after the statement of the Diagonalization Theorem.
    c. False. The 3×3 matrix in Example 4 has 3 eigenvalues, counting multiplicities, but it is not diagonalizable.
    d. False. Invertibility depends on 0 not being an eigenvalue. (See the Invertible Matrix Theorem.) A diagonalizable matrix may or may not have 0 as an eigenvalue. See Examples 3 and 5 for both possibilities.

22. a. False. The n eigenvectors must be linearly independent. See the Diagonalization Theorem.
    b. False. The matrix in Example 3 is diagonalizable, but it has only 2 distinct eigenvalues. (The statement given is the *converse* of Theorem 6.)
    c. True. This follows from $AP = PD$ and formulas (1) and (2) in the proof of the Diagonalization Theorem.
    d. False. See Example 4. The matrix there is invertible because 0 is not an eigenvalue, but the matrix is not diagonalizable.

23. A is diagonalizable because you know that five linearly independent eigenvectors exist: three in the three-dimensional eigenspace and two in the two-dimensional eigenspace. Theorem 7 guarantees that the set of all five eigenvectors is linearly independent.

24. No, by Theorem 7(b). Here is an explanation that does not appeal to Theorem 7: Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors that span the two one-dimensional eigenspaces. If $\mathbf{v}$ is any other eigenvector, then it belongs to one of the eigenspaces and hence is a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$. So there cannot exist three linearly independent eigenvectors. By the Diagonalization Theorem, A cannot be diagonalizable.

25. Let $\{\mathbf{v}_1\}$ be a basis for the one-dimensional eigenspace, let $\mathbf{v}_2$ and $\mathbf{v}_3$ form a basis for the two-dimensional eigenspace, and let $\mathbf{v}_4$ be any eigenvector in the remaining eigenspace. By Theorem 7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. Since A is 4×4, the Diagonalization Theorem shows that A is diagonalizable.

26. Yes, if the third eigenspace is only one-dimensional. In this case, the sum of the dimensions of the eigenspaces will be six, whereas the matrix is 7×7. See Theorem 7(b). An argument similar to that for Exercise 24 can also be given.

27. If A is diagonalizable, then $A = PDP^{-1}$ for some invertible P and diagonal D. Since A is invertible, 0 is not an eigenvalue of A. So the diagonal entries in D (which are eigenvalues of A) are not zero, and D is invertible. By the theorem on the inverse of a product,

$$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$$

Since D^{-1} is obviously diagonal, A^{-1} is diagonalizable.

28. If A has n linearly independent eigenvectors, then by the Diagonalization Theorem, $A = PDP^{-1}$ for some invertible P and diagonal D. Using properties of transposes,

$$A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T$$
$$= (P^T)^{-1} DP^T = QDQ^{-1}$$

where $Q = (P^T)^{-1}$. Thus A^T is diagonalizable. By the Diagonalization Theorem, the columns of Q are n linearly independent eigenvectors of A^T.

29. The diagonal entries in D_1 are reversed from those in D. So interchange the (eigenvector) columns of P to make them correspond properly to the eigenvalues in D_1. In this case,

$$P_1 = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \text{ and } D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

Although the first column of P must be an eigenvector corresponding to the eigenvalue 3, there is nothing to prevent us from selecting some multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, say $\begin{bmatrix} -3 \\ 6 \end{bmatrix}$, and letting $P_2 = \begin{bmatrix} -3 & 1 \\ 6 & -1 \end{bmatrix}$. We now have three different factorizations or "diagonalizations" of A:

$$A = PDP^{-1} = P_1 D_1 P_1^{-1} = P_2 D_1 P_2^{-1}$$

30. A nonzero multiple of an eigenvector is another eigenvector. To produce P_2, simply multiply one or both columns of P by a nonzero scalar unequal to 1.

31. For a 2×2 matrix A to be invertible, its eigenvalues must be nonzero. A first attempt at a construction might be something such as $\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, whose eigenvalues are 2 and 4. Unfortunately, a 2×2 matrix with two distinct eigenvalues is diagonalizable (Theorem 6). So, adjust the construction to $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, which works. In fact, any matrix of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ has the desired properties when a and b are nonzero. The eigenspace for the eigenvalue a is one-dimensional, as a simple calculation shows, and there is no other eigenvalue to produce a second eigenvector.

32. Any 2×2 matrix with two distinct eigenvalues is diagonalizable, by Theorem 6. If one of those eigenvalues is zero, then the matrix will not be invertible. Any matrix of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ has the desired properties when a and b are nonzero. The number a must be nonzero to make the matrix diagonalizable; b must be nonzero to make the matrix not diagonal. Other solutions are $\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$.

33. $A=\begin{bmatrix}-6 & 4 & 0 & 9\\ -3 & 0 & 1 & 6\\ -1 & -2 & 1 & 0\\ -4 & 4 & 0 & 7\end{bmatrix}$,

`ev = eig(A) = (5,1,-2,-2)`

`nulbasis(A-ev(1)*eye(4)) =` $\begin{bmatrix}1.0000\\ 0.5000\\ -0.5000\\ 1.0000\end{bmatrix}$

A basis for the eigenspace of $\lambda=5$ is $\begin{bmatrix}2\\ 1\\ -1\\ 2\end{bmatrix}$.

`nulbasis(A-ev(2)*eye(4)) =` $\begin{bmatrix}1.0000\\ -0.5000\\ -3.5000\\ 1.0000\end{bmatrix}$

A basis for the eigenspace of $\lambda=1$ is $\begin{bmatrix}2\\ -1\\ -7\\ 2\end{bmatrix}$.

`nulbasis(A-ev(3)*eye(4)) =` $\begin{bmatrix}1.0000\\ 1.0000\\ 1.0000\\ 0\end{bmatrix}, \begin{bmatrix}1.5000\\ -0.7500\\ 0\\ 1.0000\end{bmatrix}$

A basis for the eigenspace of $\lambda=-2$ is $\begin{bmatrix}1\\ 1\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}6\\ -3\\ 0\\ 4\end{bmatrix}$.

Thus we construct $P=\begin{bmatrix}2 & 2 & 1 & 6\\ 1 & -1 & 1 & -3\\ -1 & -7 & 1 & 0\\ 2 & 2 & 0 & 4\end{bmatrix}$ and $D=\begin{bmatrix}5 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}$.

34. $A=\begin{bmatrix}0 & 13 & 8 & 4\\ 4 & 9 & 8 & 4\\ 8 & 6 & 12 & 8\\ 0 & 5 & 0 & -4\end{bmatrix}$,

`ev = eig(A) = (-4,24,1,-4)`

$$\texttt{nulbasis(A-ev(1)*eye(4))} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the eigenspace of $\lambda = -4$ is $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\texttt{nulbasis(A-ev(2)*eye(4))} = \begin{bmatrix} 5.6000 \\ 5.6000 \\ 7.2000 \\ 1.0000 \end{bmatrix}$$

A basis for the eigenspace of $\lambda = 24$ is $\begin{bmatrix} 28 \\ 28 \\ 36 \\ 5 \end{bmatrix}$.

$$\texttt{nulbasis(A-ev(3)*eye(4))} = \begin{bmatrix} 1.0000 \\ 1.0000 \\ -2.0000 \\ 1.0000 \end{bmatrix}$$

A basis for the eigenspace of $\lambda = 1$ is $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$.

Thus we construct $P = \begin{bmatrix} -2 & -1 & 28 & 1 \\ 0 & 0 & 28 & 1 \\ 1 & 0 & 36 & -2 \\ 0 & 1 & 5 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

35. $A = \begin{bmatrix} 11 & -6 & 4 & -10 & -4 \\ -3 & 5 & -2 & 4 & 1 \\ -8 & 12 & -3 & 12 & 4 \\ 1 & 6 & -2 & 3 & -1 \\ 8 & -18 & 8 & -14 & -1 \end{bmatrix}$,

`ev = eig(A) = (5,1,3,5,1)`

$$\texttt{nulbasis(A-ev(1)*eye(5))} = \begin{bmatrix} 2.0000 \\ -0.3333 \\ -1.0000 \\ 1.0000 \\ 0 \end{bmatrix}, \begin{bmatrix} 1.0000 \\ -0.3333 \\ -1.0000 \\ 0 \\ 1.0000 \end{bmatrix}$$

A basis for the eigenspace of $\lambda = 5$ is $\begin{bmatrix} 6 \\ -1 \\ -3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -3 \\ 0 \\ 3 \end{bmatrix}$.

`nulbasis(A-ev(2)*eye(5))` $= \begin{bmatrix} 0.8000 \\ -0.6000 \\ -0.4000 \\ 1.0000 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.6000 \\ -0.2000 \\ -0.8000 \\ 0 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 1$ is $\begin{bmatrix} 4 \\ -3 \\ -2 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -4 \\ 0 \\ 5 \end{bmatrix}$.

`nulbasis(A-ev(3)*eye(5))` $= \begin{bmatrix} 0.5000 \\ -0.2500 \\ -1.0000 \\ -0.2500 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 3$ is $\begin{bmatrix} 2 \\ -1 \\ -4 \\ -1 \\ 4 \end{bmatrix}$.

Thus we construct $P = \begin{bmatrix} 6 & 3 & 4 & 3 & 2 \\ -1 & -1 & -3 & -1 & -1 \\ -3 & -3 & -2 & -4 & -4 \\ 3 & 0 & 5 & 0 & -1 \\ 0 & 3 & 0 & 5 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

36. $A = \begin{bmatrix} 4 & 4 & 2 & 3 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 6 & 12 & 11 & 2 & -4 \\ 9 & 20 & 10 & 10 & -6 \\ 15 & 28 & 14 & 5 & -3 \end{bmatrix}$,

`ev = eig(A) = (3,5,7,5,3)`

$$\texttt{nulbasis(A-ev(1)*eye(5))} = \begin{bmatrix} 2.0000 \\ -1.5000 \\ 0.5000 \\ 1.0000 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.0000 \\ 0.5000 \\ 0.5000 \\ 0 \\ 1.0000 \end{bmatrix}$$

A basis for the eigenspace of $\lambda = 3$ is $\begin{bmatrix} 4 \\ -3 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$.

$$\texttt{nulbasis(A-ev(2)*eye(5))} = \begin{bmatrix} 0 \\ -0.5000 \\ 1.0000 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.0000 \\ 1.0000 \\ 0 \\ -1.0000 \\ 1.0000 \end{bmatrix}$$

A basis for the eigenspace of $\lambda = 5$ is $\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

$$\texttt{nulbasis(A-ev(3)*eye(5))} = \begin{bmatrix} 0.3333 \\ 0.0000 \\ 0.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}$$

A basis for the eigenspace of $\lambda = 7$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}$.

Thus we construct $P = \begin{bmatrix} 4 & -2 & 0 & -1 & 1 \\ -3 & 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & -1 & 3 \\ 0 & 2 & 0 & 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$.

Notes: For your use, here is another matrix with five distinct real eigenvalues. To four decimal places, they are 11.0654, 9.8785, 3.8238, −3.7332, and −6.0345.

$$\begin{bmatrix} 6 & -8 & 5 & -3 & 0 \\ -7 & 3 & -5 & 3 & 0 \\ -3 & -7 & 5 & -3 & 5 \\ 0 & -4 & 1 & -7 & 5 \\ -5 & -3 & -2 & 0 & 8 \end{bmatrix}$$

The MATLAB box in the *Study Guide* encourages students to use `eig (A)` and `nulbasis` to practice the diagonalization procedure in this section. It also remarks that in later work, a student may automate the process, using the command `[P D]= eig (A)`. You may wish to permit students to use the full power of `eig` in some problems in Sections 5.5 and 5.7.