5.4 SOLUTIONS

1. Since $T(\mathbf{b}_1) = 3\mathbf{d}_1 - 5\mathbf{d}_2, [T(\mathbf{b}_1)]_D = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Likewise $T(\mathbf{b}_2) = -\mathbf{d}_1 + 6\mathbf{d}_2$ implies that $[T(\mathbf{b}_2)]_D = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ and

   $T(\mathbf{b}_3) = 4\mathbf{d}_2$ implies that $[T(\mathbf{b}_3)]_D = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$. Thus the matrix for T relative to B and

   D is $\left[[T(\mathbf{b}_1)]_D [T(\mathbf{b}_2)]_D [T(\mathbf{b}_3)]_D \right] = \begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{bmatrix}$.

2. Since $T(\mathbf{d}_1) = 2\mathbf{b}_1 - 3\mathbf{b}_2, [T(\mathbf{d}_1)]_B = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Likewise $T(\mathbf{d}_2) = -4\mathbf{b}_1 + 5\mathbf{b}_2$ implies that $[T(\mathbf{d}_2)]_B = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$.

   Thus the matrix for T relative to D and B is $\left[[T(\mathbf{d}_1)]_B [T(\mathbf{d}_2)]_B \right] = \begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$.

3. **a.** $T(\mathbf{e}_1) = 0\mathbf{b}_1 - 1\mathbf{b}_2 + \mathbf{b}_3, T(\mathbf{e}_2) = -1\mathbf{b}_1 - 0\mathbf{b}_2 - 1\mathbf{b}_3, T(\mathbf{e}_3) = 1\mathbf{b}_1 - 1\mathbf{b}_2 + 0\mathbf{b}_3$

   b. $[T(\mathbf{e}_1)]_B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, [T(\mathbf{e}_2)]_B = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, [T(\mathbf{e}_3)]_B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

   c. The matrix for T relative to $\mathcal{E}$ and B is $[\ [T(\mathbf{e}_1)]_B \quad [T(\mathbf{e}_2)]_B \quad [T(\mathbf{e}_3)]_B] = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

4. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$. Since $[T(\mathbf{b}_1)]_{\mathcal{E}} = T(\mathbf{b}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, [T(\mathbf{b}_2)]_{\mathcal{E}} = T(\mathbf{b}_2) = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$,

   and $[T(\mathbf{b}_3)]_{\mathcal{E}} = T(\mathbf{b}_3) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, the matrix for T relative to B and $\mathcal{E}$ is $[[T(\mathbf{b}_1)]_{\mathcal{E}} \quad [T(\mathbf{b}_2)]_{\mathcal{E}} \quad [T(\mathbf{b}_3)]_{\mathcal{E}}] =$

   $\begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}$.

5. **a**. $T(\mathbf{p}) = (t+5)(2-t+t^2) = 10-3t+4t^2+t^3$

b. Let $\mathbf{p}$ and $\mathbf{q}$ be polynomials in $\mathbb{P}_2$, and let c be any scalar. Then

$$\begin{aligned} T(\mathbf{p}(t)+\mathbf{q}(t)) &= (t+5)[\mathbf{p}(t)+\mathbf{q}(t)] = (t+5)\mathbf{p}(t)+(t+5)\mathbf{q}(t) \\ &= T(\mathbf{p}(t))+T(\mathbf{q}(t)) \\ T(c\cdot\mathbf{p}(t)) &= (t+5)[c\cdot\mathbf{p}(t)] = c\cdot(t+5)\mathbf{p}(t) \\ &= c\cdot T[\mathbf{p}(t)] \end{aligned}$$

and T is a linear transformation.

c. Let $B=\{1,t,t^2\}$ and $C=\{1,t,t^2,t^3\}$. Since $T(\mathbf{b}_1)=T(1)=(t+5)(1)=t+5$, $[T(\mathbf{b}_1)]_C = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Likewise

since $T(\mathbf{b}_2)=T(t)=(t+5)(t)=t^2+5t$, $[T(\mathbf{b}_2)]_C = \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}$, and since

$T(\mathbf{b}_3)=T(t^2)=(t+5)(t^2)=t^3+5t^2$, $[T(\mathbf{b}_3)]_C = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 1 \end{bmatrix}$. Thus the matrix for T relative to B and

C is $[\,[T(\mathbf{b}_1)]_C \quad [T(\mathbf{b}_2)]_C \quad [T(\mathbf{b}_3)]_C\,] = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$.

6. **a**. $T(\mathbf{p}) = (2-t+t^2)+t^2(2-t+t^2) = 2-t+3t^2-t^3+t^4$

b. Let $\mathbf{p}$ and $\mathbf{q}$ be polynomials in $\mathbb{P}_2$, and let c be any scalar. Then

$$\begin{aligned} T(\mathbf{p}(t)+\mathbf{q}(t)) &= [\mathbf{p}(t)+\mathbf{q}(t)]+t^2[\mathbf{p}(t)+\mathbf{q}(t)] \\ &= [\mathbf{p}(t)+t^2\mathbf{p}(t)]+[\mathbf{q}(t)+t^2\mathbf{q}(t)] \\ &= T(\mathbf{p}(t))+T(\mathbf{q}(t)) \\ T(c\cdot\mathbf{p}(t)) &= [c\cdot\mathbf{p}(t)]+t^2[c\cdot\mathbf{p}(t)] \\ &= c\cdot[\mathbf{p}(t)+t^2\mathbf{p}(t)] \\ &= c\cdot T[\mathbf{p}(t)] \end{aligned}$$

and T is a linear transformation.

c. Let $B=\{1,t,t^2\}$ and $C=\{1,t,t^2,t^3,t^4\}$. Since $T(\mathbf{b}_1)=T(1)=1+t^2(1)=t^2+1, [T(\mathbf{b}_1)]_C=\begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix}$.

Likewise since $T(\mathbf{b}_2)=T(t)=t+(t^2)(t)=t^3+t, [T(\mathbf{b}_2)]_C=\begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}$, and

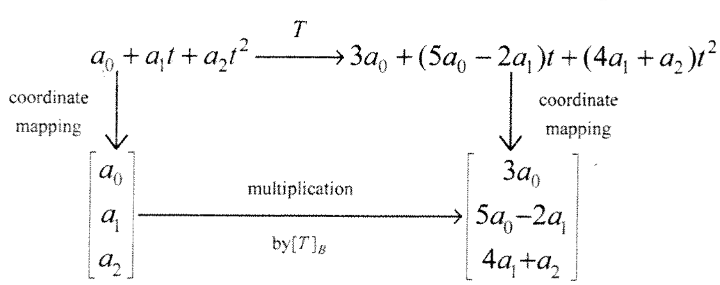
since $T(\mathbf{b}_3)=T(t^2)=t^2+(t^2)(t^2)=t^4+t^2, [T(\mathbf{b}_3)]_C=\begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix}$. Thus the matrix for T relative to

B and C is $[\ [T(\mathbf{b}_1)]_C\ \ [T(\mathbf{b}_2)]_C\ \ [T(\mathbf{b}_3)]_C]=\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\\0&1&0\\0&0&1\end{bmatrix}$.

7. Since $T(\mathbf{b}_1)=T(1)=3+5t, [T(\mathbf{b}_1)]_B=\begin{bmatrix}3\\5\\0\end{bmatrix}$. Likewise since $T(\mathbf{b}_2)=T(t)=-2t+4t^2, [T(\mathbf{b}_2)]_B=\begin{bmatrix}0\\-2\\4\end{bmatrix}$,

and since $T(\mathbf{b}_3)=T(t^2)=t^2, [T(\mathbf{b}_3)]_B=\begin{bmatrix}0\\0\\1\end{bmatrix}$. Thus the matrix representation of T relative to the basis

B is $\left[[T(\mathbf{b}_1)]_B\ \ [T(\mathbf{b}_2)]_B\ \ [T(\mathbf{b}_3)]_B\right]=\begin{bmatrix}3&0&0\\5&-2&0\\0&4&1\end{bmatrix}$. Perhaps a faster way is to realize that the information given provides the general form of $T(\mathbf{p})$ as shown in the figure below:



The matrix that implements the multiplication along the bottom of the figure is easily filled in by inspection:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3a_0 \\ 5a_0-2a_1 \\ 4a_1+a_2 \end{bmatrix} \text{ implies that } [T]_B = \begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

8. Since $[3\mathbf{b}_1 - 4\mathbf{b}_2]_B = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$, $[T(3\mathbf{b}_1 - 4\mathbf{b}_2)]_B = [T]_B[3\mathbf{b}_1 - 4\mathbf{b}_2]_B = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ -20 \\ 11 \end{bmatrix}$

and $T(3\mathbf{b}_1 - 4\mathbf{b}_2) = 24\mathbf{b}_1 - 20\mathbf{b}_2 + 11\mathbf{b}_3$.

9. a. $T(\mathbf{p}) = \begin{bmatrix} 5+3(-1) \\ 5+3(0) \\ 5+3(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

b. Let $\mathbf{p}$ and $\mathbf{q}$ be polynomials in $\mathbb{P}_2$, and let c be any scalar. Then

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(-1) \\ (\mathbf{p}+\mathbf{q})(0) \\ (\mathbf{p}+\mathbf{q})(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-1)+\mathbf{q}(-1) \\ \mathbf{p}(0)+\mathbf{q}(0) \\ \mathbf{p}(1)+\mathbf{q}(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(-1) \\ \mathbf{q}(0) \\ \mathbf{q}(1) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q})$$

$$T(c\cdot\mathbf{p}) = \begin{bmatrix} (c\cdot\mathbf{p})(-1) \\ (c\cdot\mathbf{p})(0) \\ (c\cdot\mathbf{p})(1) \end{bmatrix} = \begin{bmatrix} c\cdot(\mathbf{p}(-1)) \\ c\cdot(\mathbf{p}(0)) \\ c\cdot(\mathbf{p}(1)) \end{bmatrix} = c\cdot\begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = c\cdot T(\mathbf{p})$$

and T is a linear transformation.

c. Let $B = \{1, t, t^2\}$ and $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Since

$$[T(\mathbf{b}_1)]_{\mathcal{E}} = T(\mathbf{b}_1) = T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ [T(\mathbf{b}_2)]_{\mathcal{E}} = T(\mathbf{b}_2) = T(t) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } [T(\mathbf{b}_3)]_{\mathcal{E}} = T(\mathbf{b}_3) = T(t^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

the matrix for T relative to B and $\mathcal{E}$ is $\left[[T(\mathbf{b}_1)]_{\mathcal{E}}\ \ [T(\mathbf{b}_2)]_{\mathcal{E}}\ \ [T(\mathbf{b}_3)]_{\mathcal{E}}\right] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

10. a. Let $\mathbf{p}$ and $\mathbf{q}$ be polynomials in $\mathbb{P}_3$, and let c be any scalar. Then

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(-3) \\ (\mathbf{p}+\mathbf{q})(-1) \\ (\mathbf{p}+\mathbf{q})(1) \\ (\mathbf{p}+\mathbf{q})(3) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-3)+\mathbf{q}(-3) \\ \mathbf{p}(-1)+\mathbf{q}(-1) \\ \mathbf{p}(1)+\mathbf{q}(1) \\ \mathbf{p}(3)+\mathbf{q}(3) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-3) \\ \mathbf{p}(-1) \\ \mathbf{p}(1) \\ \mathbf{p}(3) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(-3) \\ \mathbf{q}(-1) \\ \mathbf{q}(1) \\ \mathbf{q}(3) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q})$$

$$T(c\cdot\mathbf{p}) = \begin{bmatrix} (c\cdot\mathbf{p})(-3) \\ (c\cdot\mathbf{p})(-1) \\ (c\cdot\mathbf{p})(1) \\ (c\cdot\mathbf{p})(3) \end{bmatrix} = \begin{bmatrix} c\cdot(\mathbf{p}(-3)) \\ c\cdot(\mathbf{p}(-1)) \\ c\cdot(\mathbf{p}(1)) \\ c\cdot(\mathbf{p}(3)) \end{bmatrix} = c\cdot\begin{bmatrix} \mathbf{p}(-3) \\ \mathbf{p}(-1) \\ \mathbf{p}(1) \\ \mathbf{p}(3) \end{bmatrix} = c\cdot T(\mathbf{p})$$

and T is a linear transformation.

b. Let $B = \{1, t, t^2, t^3\}$ and $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis for $\mathbb{R}^3$. Since

$$[T(\mathbf{b}_1)]_{\mathcal{E}} = T(\mathbf{b}_1) = T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ [T(\mathbf{b}_2)]_{\mathcal{E}} = T(\mathbf{b}_2) = T(t) = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \ [T(\mathbf{b}_3)]_{\mathcal{E}} = T(\mathbf{b}_3) = T(t^2) = \begin{bmatrix} 9 \\ 1 \\ 1 \\ 9 \end{bmatrix}, \text{ and}$$

$$[T(\mathbf{b}_4)]_{\mathcal{E}} = T(\mathbf{b}_4) = T(t^3) = \begin{bmatrix} -27 \\ -1 \\ 1 \\ 27 \end{bmatrix}, \text{ the matrix for } T \text{ relative to } B \text{ and } \mathcal{E} \text{ is}$$

$$\begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{E}} & [T(\mathbf{b}_2)]_{\mathcal{E}} & [T(\mathbf{b}_3)]_{\mathcal{E}} & [T(\mathbf{b}_4)]_{\mathcal{E}} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9 & -27 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix}.$$

11. Following Example 4, if $P = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, then the B-matrix is

$$P^{-1}AP = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

12. Following Example 4, if $P = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$, then the B-matrix is

$$P^{-1}AP = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

13. Start by diagonalizing A. The characteristic polynomial is $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$, so the eigenvalues of A are 1 and 3.

<u>For $\lambda = 1$:</u> $A - I = \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}$. The equation $(A - I)\mathbf{x} = \mathbf{0}$ amounts to $-x_1 + x_2 = 0$, so $x_1 = x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

<u>For $\lambda = 3$:</u> $A - 3I = \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}$. The equation $(A - 3I)\mathbf{x} = \mathbf{0}$ amounts to $-3x_1 + x_2 = 0$, so $x_1 = (1/3)x_2$ with x_2 free. A nice basis vector for the eigenspace is thus $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ we may construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ which diagonalizes A. By Theorem 8, the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ has the property that the B-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a diagonal matrix.

14. Start by diagonalizing A. The characteristic polynomial is $\lambda^2 - 6\lambda - 16 = (\lambda - 8)(\lambda + 2)$, so the eigenvalues of A are 8 and -2.

For $\lambda = 8$: $A - 8I = \begin{bmatrix} -3 & -3 \\ -7 & -7 \end{bmatrix}$. The equation $(A - 8I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 + x_2 = 0$, so $x_1 = -x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

For $\lambda = 2$: $A + 2I = \begin{bmatrix} 7 & -3 \\ -7 & 3 \end{bmatrix}$. The equation $(A - 2I)\mathbf{x} = \mathbf{0}$ amounts to $7x_1 - 3x_2 = 0$, so $x_1 = (3/7)x_2$ with x_2 free. A nice basis vector for the eigenspace is thus $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ we may construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & 7 \end{bmatrix}$ which diagonalizes A. By Theorem 8, the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ has the property that the B-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a diagonal matrix.

15. Start by diagonalizing A. The characteristic polynomial is $\lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2)$, so the eigenvalues of A are 5 and 2.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}$. The equation $(A - 5I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 + 2x_2 = 0$, so $x_1 = -2x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$. The equation $(A - 2I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 - x_2 = 0$, so $x_1 = x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ we may construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ which diagonalizes A. By Theorem 8, the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ has the property that the B-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a diagonal matrix.

16. Start by diagonalizing A. The characteristic polynomial is $\lambda^2 - 5\lambda = \lambda(\lambda - 5)$, so the eigenvalues of A are 5 and 0.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}$. The equation $(A - 5I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 + 2x_2 = 0$, so $x_1 = -2x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

For $\lambda = 0$: $A - 0I = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$. The equation $(A - 0I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 - 3x_2 = 0$, so $x_1 = 3x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ we may construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}=\begin{bmatrix}-2 & 3\\ 1 & 1\end{bmatrix}$ which diagonalizes A. By Theorem 8, the basis $B=\{\mathbf{v}_1,\mathbf{v}_2\}$ has the property that the B-matrix of the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is a diagonal matrix.

17. **a**. We compute that

$$A\mathbf{b}_1=\begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}2\\ 2\end{bmatrix}=2\mathbf{b}_1$$

so $\mathbf{b}_1$ is an eigenvector of A corresponding to the eigenvalue 2. The characteristic polynomial of A is $\lambda^2-4\lambda+4=(\lambda-2)^2$, so 2 is the only eigenvalue for A. Now $A-2I=\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}$, which implies that the eigenspace corresponding to the eigenvalue 2 is one-dimensional. Thus the matrix A is not diagonalizable.

b. Following Example 4, if $P=\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}$, then the B-matrix for T is

$$P^{-1}AP=\begin{bmatrix}-4 & 5\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix}=\begin{bmatrix}1 & 5\\ 1 & 4\end{bmatrix}=\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}$$

18. If there is a basis B such that $[T]_B$ is diagonal, then A is similar to a diagonal matrix, by the second paragraph following Example 3. In this case, A would have three linearly independent eigenvectors. However, this is not necessarily the case, because A has only two distinct eigenvalues.

19. If A is similar to B, then there exists an invertible matrix P such that $P^{-1}AP=B$. Thus B is invertible because it is the product of invertible matrices. By a theorem about inverses of products, $B^{-1}=P^{-1}A^{-1}(P^{-1})^{-1}=P^{-1}A^{-1}P$, which shows that A^{-1} is similar to B^{-1}.

20. If $A=PBP^{-1}$, then $A^2=(PBP^{-1})(PBP^{-1})=PB(P^{-1}P)BP^{-1}=PB\cdot I\cdot BP^{-1}=PB^2P^{-1}$. So A^2 is similar to B^2.

21. By hypothesis, there exist invertible P and Q such that $P^{-1}BP=A$ and $Q^{-1}CQ=A$. Then $P^{-1}BP=Q^{-1}CQ$. Left-multiply by Q and right-multiply by Q^{-1} to obtain $QP^{-1}BPQ^{-1}=QQ^{-1}CQQ^{-1}$. So $C=QP^{-1}BPQ^{-1}=(PQ^{-1})^{-1}B(PQ^{-1})$, which shows that B is similar to C.

22. If A is diagonalizable, then $A=PDP^{-1}$ for some P. Also, if B is similar to A, then $B=QAQ^{-1}$ for some Q. Then $B=Q(PDP^{-1})Q^{-1}=(QP)D(P^{-1}Q^{-1})=(QP)D(QP)^{-1}$
So B is diagonalizable.

23. If $A\mathbf{x}=\lambda\mathbf{x}$, $\mathbf{x}\neq 0$, then $P^{-1}A\mathbf{x}=\lambda P^{-1}\mathbf{x}$. If $B=P^{-1}AP$, then

$$B(P^{-1}\mathbf{x})=P^{-1}AP(P^{-1}\mathbf{x})=P^{-1}A\mathbf{x}=\lambda P^{-1}\mathbf{x} \qquad (*)$$

by the first calculation. Note that $P^{-1}\mathbf{x}\neq 0$, because $\mathbf{x}\neq 0$ and P^{-1} is invertible. Hence (*) shows that $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to λ. (Of course, λ is an eigenvalue of both A and B because the matrices are similar, by Theorem 4 in Section 5.2.)

24. If $A=PBP^{-1}$, then $\text{rank}\,A=\text{rank}\,P(BP^{-1})=\text{rank}\,BP^{-1}$, by Supplementary Exercise 13 in Chapter 4. Also, $\text{rank}\,BP^{-1}=\text{rank}\,B$, by Supplementary Exercise 14 in Chapter 4, since P^{-1} is invertible. Thus $\text{rank}\,A=\text{rank}\,B$.

25. If $A = PBP^{-1}$, then

$$\text{tr}(A) = \text{tr}((PB)P^{-1}) = \text{tr}(P^{-1}(PB)) \qquad \text{By the trace property}$$
$$= \text{tr}(P^{-1}PB) = \text{tr}(IB) = \text{tr}(B)$$

If B is diagonal, then the diagonal entries of B must be the eigenvalues of A, by the Diagonalization Theorem (Theorem 5 in Section 5.3). So $\text{tr } A = \text{tr } B = \{\text{sum of the eigenvalues of } A\}$.

26. If $A = PDP^{-1}$ for some P, then the general trace property from Exercise 25 shows that $\text{tr } A = \text{tr } [(PD)P^{-1}] = \text{tr } [P^{-1}PD] = \text{tr } D$. (Or, one can use the result of Exercise 25 that since A is similar to D, $\text{tr } A = \text{tr } D$.) Since the eigenvalues of A are on the main diagonal of D, tr D is the sum of the eigenvalues of A.

27. For each j, $I(\mathbf{b}_j) = \mathbf{b}_j$. Since the standard coordinate vector of any vector in $\mathbb{R}^n$ is just the vector itself, $[I(\mathbf{b}_j)]_{\mathcal{E}} = \mathbf{b}_j$. Thus the matrix for I relative to B and the standard basis $\mathcal{E}$ is simply $\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$. This matrix is precisely the *change-of-coordinates* matrix P_B defined in Section 4.4.

28. For each j, $I(\mathbf{b}_j) = \mathbf{b}_j$, and $[I(\mathbf{b}_j)]_C = [\mathbf{b}_j]_C$. By formula (4), the matrix for I relative to the bases B and C is

$$M = \begin{bmatrix} [\mathbf{b}_1]_C & [\mathbf{b}_2]_C & \cdots & [\mathbf{b}_n]_C \end{bmatrix}$$

In Theorem 15 of Section 4.7, this matrix was denoted by $\underset{C \leftarrow B}{P}$ and was called the *change-of-coordinates matrix from* B *to* C.

29. If $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then the B-coordinate vector of $\mathbf{b}_j$ is $\mathbf{e}_j$, the standard basis vector for $\mathbb{R}^n$. For instance,

$$\mathbf{b}_1 = 1 \cdot \mathbf{b}_1 + 0 \cdot \mathbf{b}_2 + \cdots + 0 \cdot \mathbf{b}_n$$

Thus $[I(\mathbf{b}_j)]_B = [\mathbf{b}_j]_B = \mathbf{e}_j$, and

$$[I]_B = \begin{bmatrix} [I(\mathbf{b}_1)]_B \cdots [I(\mathbf{b}_n)]_B \end{bmatrix} = [\mathbf{e}_1 \cdots \mathbf{e}_n] = I$$

30. **[M]** If P is the matrix whose columns come from B, then the B-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $D = P^{-1}AP$. From the data in the text,

$$A = \begin{bmatrix} -14 & 4 & -14 \\ -33 & 9 & -31 \\ 11 & -4 & 11 \end{bmatrix}, P = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ -2 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -14 & 4 & -14 \\ -33 & 9 & -31 \\ 11 & -4 & 11 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ -2 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 3 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{bmatrix}$$

31. **[M]** If P is the matrix whose columns come from B, then the B-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $D = P^{-1}AP$. From the data in the text,

$$A = \begin{bmatrix} -7 & -48 & -16 \\ 1 & 14 & 6 \\ -3 & -45 & -19 \end{bmatrix}, P = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} -3 & -2 & 3 \\ 1 & 1 & -1 \\ -3 & -3 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & -3 & -1/3 \\ 1 & 3 & 0 \\ 0 & -1 & -1/3 \end{bmatrix}\begin{bmatrix} -7 & -48 & -16 \\ 1 & 14 & 6 \\ -3 & -45 & -19 \end{bmatrix}\begin{bmatrix} -3 & -2 & 3 \\ 1 & 1 & -1 \\ -3 & -3 & 0 \end{bmatrix} = \begin{bmatrix} -7 & -2 & -6 \\ 0 & -4 & -6 \\ 0 & 0 & -1 \end{bmatrix}$$

32. **[M]** $A = \begin{bmatrix} 15 & -66 & -44 & -33 \\ 0 & 13 & 21 & -15 \\ 1 & -15 & -21 & 12 \\ 2 & -18 & -22 & 8 \end{bmatrix}$,

`ev=eig(A)` = (2, 4, 4, 5)

`nulbasis(A-ev(1)*eye(4))` = $\begin{bmatrix} 0.0000 \\ -1.5000 \\ 1.5000 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 2$ is $\mathbf{b}_1 = \begin{bmatrix} 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}$.

`nulbasis(A-ev(2)*eye(4))` = $\begin{bmatrix} -10.0000 \\ -2.3333 \\ 1.0000 \\ 0 \end{bmatrix}, \begin{bmatrix} 13.0000 \\ 1.6667 \\ 0 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 4$ is $\{\mathbf{b}_2, \mathbf{b}_3\} = \left\{ \begin{bmatrix} -30 \\ -7 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 39 \\ 5 \\ 0 \\ 3 \end{bmatrix} \right\}$.

`nulbasis(A-ev(4)*eye(4))` = $\begin{bmatrix} 2.7500 \\ -0.7500 \\ 1.0000 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 5$ is $\mathbf{b}_4 = \begin{bmatrix} 11 \\ -3 \\ 4 \\ 4 \end{bmatrix}$.

The basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ is a basis for $\mathbb{R}^4$ with the property that $[T]_B$ is diagonal.