6 Orthogonality and Least Squares

6.1 SOLUTIONS

Notes: The first half of this section is computational and is easily learned. The second half concerns the concepts of orthogonality and orthogonal complements, which are essential for later work. Theorem 3 is an important general fact, but is needed only for Supplementary Exercise 13 at the end of the chapter and in Section 7.4. The optional material on angles is not used later. Exercises 27–31 concern facts used later.

1. Since $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $\mathbf{u} \cdot \mathbf{u} = (-1)^2 + 2^2 = 5$, $\mathbf{v} \cdot \mathbf{u} = 4(-1) + 6(2) = 8$, and $\dfrac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} = \dfrac{8}{5}$.

2. Since $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{w} \cdot \mathbf{w} = 3^2 + (-1)^2 + (-5)^2 = 35$, $\mathbf{x} \cdot \mathbf{w} = 6(3) + (-2)(-1) + 3(-5) = 5$, and
$\dfrac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} = \dfrac{5}{35} = \dfrac{1}{7}$.

3. Since $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$, $\mathbf{w} \cdot \mathbf{w} = 3^2 + (-1)^2 + (-5)^2 = 35$, and $\dfrac{1}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w} = \begin{bmatrix} 3/35 \\ -1/35 \\ -1/7 \end{bmatrix}$.

4. Since $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{u} \cdot \mathbf{u} = (-1)^2 + 2^2 = 5$ and $\dfrac{1}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$.

5. Since $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $\mathbf{u} \cdot \mathbf{v} = (-1)(4) + 2(6) = 8$, $\mathbf{v} \cdot \mathbf{v} = 4^2 + 6^2 = 52$, and
$\left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v} = \dfrac{2}{13}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix}$.

6. Since $\mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$, $\mathbf{x} \cdot \mathbf{w} = 6(3) + (-2)(-1) + 3(-5) = 5$, $\mathbf{x} \cdot \mathbf{x} = 6^2 + (-2)^2 + 3^2 = 49$, and
$\left(\dfrac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{x} \cdot \mathbf{x}}\right)\mathbf{x} = \dfrac{5}{49}\begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix}$.

7. Since $\mathbf{w}=\begin{bmatrix}3\\-1\\-5\end{bmatrix}$, $\|\mathbf{w}\|=\sqrt{\mathbf{w}\cdot\mathbf{w}}=\sqrt{3^2+(-1)^2+(-5)^2}=\sqrt{35}.$

8. Since $\mathbf{x}=\begin{bmatrix}6\\-2\\3\end{bmatrix}$, $\|\mathbf{x}\|=\sqrt{\mathbf{x}\cdot\mathbf{x}}=\sqrt{6^2+(-2)^2+3^2}=\sqrt{49}=7.$

9. A unit vector in the direction of the given vector is

$$\frac{1}{\sqrt{(-30)^2+40^2}}\begin{bmatrix}-30\\40\end{bmatrix}=\frac{1}{50}\begin{bmatrix}-30\\40\end{bmatrix}=\begin{bmatrix}-3/5\\4/5\end{bmatrix}$$

10. A unit vector in the direction of the given vector is

$$\frac{1}{\sqrt{(-6)^2+4^2+(-3)^2}}\begin{bmatrix}-6\\4\\-3\end{bmatrix}=\frac{1}{\sqrt{61}}\begin{bmatrix}-6\\4\\-3\end{bmatrix}=\begin{bmatrix}-6/\sqrt{61}\\4/\sqrt{61}\\-3\sqrt{61}\end{bmatrix}$$

11. A unit vector in the direction of the given vector is

$$\frac{1}{\sqrt{(7/4)^2+(1/2)^2+1^2}}\begin{bmatrix}7/4\\1/2\\1\end{bmatrix}=\frac{1}{\sqrt{69/16}}\begin{bmatrix}7/4\\1/2\\1\end{bmatrix}=\begin{bmatrix}7/\sqrt{69}\\2/\sqrt{69}\\4/\sqrt{69}\end{bmatrix}$$

12. A unit vector in the direction of the given vector is

$$\frac{1}{\sqrt{(8/3)^2+2^2}}\begin{bmatrix}8/3\\2\end{bmatrix}=\frac{1}{\sqrt{100/9}}\begin{bmatrix}8/3\\2\end{bmatrix}=\begin{bmatrix}4/5\\3/5\end{bmatrix}$$

13. Since $\mathbf{x}=\begin{bmatrix}10\\-3\end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix}-1\\-5\end{bmatrix}$, $\|\mathbf{x}-\mathbf{y}\|^2=[10-(-1)]^2+[-3-(-5)]^2=125$ and $\text{dist}\,(\mathbf{x},\mathbf{y})=\sqrt{125}=5\sqrt{5}.$

14. Since $\mathbf{u}=\begin{bmatrix}0\\-5\\2\end{bmatrix}$ and $\mathbf{z}=\begin{bmatrix}-4\\-1\\8\end{bmatrix}$, $\|\mathbf{u}-\mathbf{z}\|^2=[0-(-4)]^2+[-5-(-1)]^2+[2-8]^2=68$ and $\text{dist}\,(\mathbf{u},\mathbf{z})=\sqrt{68}=2\sqrt{17}.$

15. Since $\mathbf{a}\cdot\mathbf{b}=8(-2)+(-5)(-3)=-1\neq 0$, **a** and **b** are not orthogonal.

16. Since $\mathbf{u}\cdot\mathbf{v}=12(2)+(3)(-3)+(-5)(3)=0$, **u** and **v** are orthogonal.

17. Since $\mathbf{u}\cdot\mathbf{v}=3(-4)+2(1)+(-5)(-2)+0(6)=0$, **u** and **v** are orthogonal.

18. Since $\mathbf{y}\cdot\mathbf{z}=(-3)(1)+7(-8)+4(15)+0(-7)=1\neq 0$, **y** and **z** are not orthogonal.

19. **a.** True. See the definition of $\|\mathbf{v}\|$.

b. True. See Theorem 1(c).

c. True. See the discussion of Figure 5.

d. False. Counterexample: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

e. True. See the box following Example 6.

20. **a**. True. See Example 1 and Theorem 1(a).

b. False. The absolute value sign is missing. See the box before Example 2.

c. True. See the defintion of orthogonal complement.

d. True. See the Pythagorean Theorem.

e. True. See Theorem 3.

21. Theorem 1(b):

$$(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = (\mathbf{u}+\mathbf{v})^T\mathbf{w} = (\mathbf{u}^T+\mathbf{v}^T)\mathbf{w} = \mathbf{u}^T\mathbf{w}+\mathbf{v}^T\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$$

The second and third equalities used Theorems 3(b) and 2(c), respectively, from Section 2.1.

Theorem 1(c):

$$(c\mathbf{u})\cdot\mathbf{v} = (c\mathbf{u})^T\mathbf{v} = c(\mathbf{u}^T\mathbf{v}) = c(\mathbf{u}\cdot\mathbf{v})$$

The second and third equalities used Theorems 3(c) and 2(d), respectively, from Section 2.1.

22. Since $\mathbf{u}\cdot\mathbf{u}$ is the sum of the squares of the entries in $\mathbf{u}$, $\mathbf{u}\cdot\mathbf{u} \ge 0$. The sum of squares of numbers is zero if and only if all the numbers are themselves zero.

23. One computes that $\mathbf{u}\cdot\mathbf{v} = 2(-7) + (-5)(-4) + (-1)6 = 0$, $\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u} = 2^2 + (-5)^2 + (-1)^2 = 30$, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v} = (-7)^2 + (-4)^2 + 6^2 = 101$, and $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) =$ $(2+(-7))^2 + (-5+(-4))^2 + (-1+6)^2 = 131$.

24. One computes that

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$$

and

$$\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$$

so

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

25. When $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, the set H of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\boldsymbol{v}$ is the subspace of vectors whose entries satisfy $ax + by = 0$. If $a \ne 0$, then $x = -(b/a)y$ with y a free variable, and H is a line through the origin. A natural choice for a basis for H in this case is $\left\{\begin{bmatrix} -b \\ a \end{bmatrix}\right\}$. If $a = 0$ and $b \ne 0$, then $by = 0$. Since $b \ne 0$, $y = 0$ and x is a free variable. The subspace H is again a line through the origin. A natural choice for a basis for H in this case is $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, but $\left\{\begin{bmatrix} -b \\ a \end{bmatrix}\right\}$ is still a basis for H since $a = 0$ and $b \ne 0$. If $a = 0$ and $b = 0$, then $H = \mathbb{R}^2$ since the equation $0x + 0y = 0$ places no restrictions on x or y.

26. Theorem 2 in Chapter 4 may be used to show that W is a subspace of $\mathbb{R}^3$, because W is the null space of the 1×3 matrix $\mathbf{u}^T$. Geometrically, W is a plane through the origin.

27. If $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{y}\cdot\mathbf{u} = \mathbf{y}\cdot\mathbf{v} = 0$, and hence by a property of the inner product, $\mathbf{y}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{y}\cdot\mathbf{u} + \mathbf{y}\cdot\mathbf{v} = 0 + 0 = 0$. Thus $\mathbf{y}$ is orthogonal to $\mathbf{u}+\mathbf{v}$.

28. An arbitrary $\mathbf{w}$ in Span$\{\mathbf{u}, \mathbf{v}\}$ has the form $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$. If $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u}\cdot\mathbf{y} = \mathbf{v}\cdot\mathbf{y} = 0$. By Theorem 1(b) and 1(c),

$$\mathbf{w}\cdot\mathbf{y} = (c_1\mathbf{u} + c_2\mathbf{v})\cdot\mathbf{y} = c_1(\mathbf{u}\cdot\mathbf{y}) + c_2(\mathbf{v}\cdot\mathbf{y}) = 0 + 0 = 0$$

29. A typical vector in W has the form $\mathbf{w} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$. If $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$, then by Theorems 1(b) and 1(c),

$$\mathbf{w}\cdot\mathbf{x} = (c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p)\cdot\mathbf{y} = c_1(\mathbf{v}_1\cdot\mathbf{x}) + \ldots + c_p(\mathbf{v}_p\cdot\mathbf{x}) = 0$$

So $\mathbf{x}$ is orthogonal to each $\mathbf{w}$ in W.

30. **a.** If $\mathbf{z}$ is in $W^\perp$, $\mathbf{u}$ is in W, and c is any scalar, then $(c\mathbf{z})\cdot\mathbf{u} = c(\mathbf{z}\cdot\mathbf{u}) - c0 = 0$. Since $\mathbf{u}$ is any element of W, $c\mathbf{z}$ is in $W^\perp$.

b. Let $\mathbf{z}_1$ and $\mathbf{z}_2$ be in $W^\perp$. Then for any $\mathbf{u}$ in W, $(\mathbf{z}_1 + \mathbf{z}_2)\cdot\mathbf{u} = \mathbf{z}_1\cdot\mathbf{u} + \mathbf{z}_2\cdot\mathbf{u} = 0 + 0 = 0$. Thus $\mathbf{z}_1 + \mathbf{z}_2$ is in $W^\perp$.

c. Since $\mathbf{0}$ is orthogonal to every vector, $\mathbf{0}$ is in $W^\perp$. Thus $W^\perp$ is a subspace.

31. Suppose that $\mathbf{x}$ is in W and $W^\perp$. Since $\mathbf{x}$ is in $W^\perp$, $\mathbf{x}$ is orthogonal to every vector in W, including $\mathbf{x}$ itself. So $\mathbf{x}\cdot\mathbf{x} = 0$, which happens only when $\mathbf{x} = \mathbf{0}$.

32. **[M]**

a. One computes that $\|\mathbf{a}_1\| = \|\mathbf{a}_2\| = \|\mathbf{a}_3\| = \|\mathbf{a}_4\| = 1$ and that $\mathbf{a}_i\cdot\mathbf{a}_j = 0$ for $i \neq j$.

b. Answers will vary, but it should be that $\|A\mathbf{u}\| = \|\mathbf{u}\|$ and $\|A\mathbf{v}\| = \|\mathbf{v}\|$.

c. Answers will again vary, but the cosines should be equal.

d. A conjecture is that multiplying by A does not change the lengths of vectors or the angles between vectors.

33. **[M]** Answers to the calculations will vary, but will demonstrate that the mapping $\mathbf{x} \mapsto T(\mathbf{x}) = \left(\dfrac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$ (for $\mathbf{v} \neq \mathbf{0}$) is a linear transformation. To confirm this, let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$, and let c be any scalar. Then

$$T(\mathbf{x}+\mathbf{y}) = \left(\frac{(\mathbf{x}+\mathbf{y})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{(\mathbf{x}\cdot\mathbf{v}) + (\mathbf{y}\cdot\mathbf{v})}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} + \left(\frac{\mathbf{y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(c\mathbf{x}) = \left(\frac{(c\mathbf{x})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{c(\mathbf{x}\cdot\mathbf{v})}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = c\left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = cT(\mathbf{x})$$

34. **[M]** One finds that

$$N = \begin{bmatrix} -5 & 1 \\ -1 & 4 \\ 1 & 0 \\ 0 & -1 \\ 0 & 3 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 5 & 0 & -1/3 \\ 0 & 1 & 1 & 0 & -4/3 \\ 0 & 0 & 0 & 1 & 1/3 \end{bmatrix}$$

The row-column rule for computing RN produces the 3×2 zero matrix, which shows that the rows of R are orthogonal to the columns of N. This is expected by Theorem 3 since each row of R is in Row A and each column of N is in Nul A.