6.2 SOLUTIONS

Notes: The nonsquare matrices in Theorems 6 and 7 are needed for the QR factorizarion in Section 6.4. It is important to emphasize that the term *orthogonal matrix* applies only to certain *square* matrices. The subsection on orthogonal projections not only sets the stage for the general case in Section 6.3, it also provides what is needed for the orthogonal diagonalization exercises in Section 7.1, because none of the eigenspaces there have dimension greater than 2. For this reason, the Gram-Schmidt process (Section 6.4) is not really needed in Chapter 7. Exercises 13 and 14 prepare for Section 6.3.

1. Since $\begin{bmatrix}-1\\4\\-3\end{bmatrix}\cdot\begin{bmatrix}3\\-4\\-7\end{bmatrix}=2\neq 0,$ the set is not orthogonal.

2. Since $\begin{bmatrix}1\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\2\end{bmatrix}=\begin{bmatrix}1\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}-5\\-2\\1\end{bmatrix}=\begin{bmatrix}0\\1\\2\end{bmatrix}\cdot\begin{bmatrix}-5\\-2\\1\end{bmatrix}=0,$ the set is orthogonal.

3. Since $\begin{bmatrix}-6\\-3\\9\end{bmatrix}\cdot\begin{bmatrix}3\\1\\-1\end{bmatrix}=-30\neq 0,$ the set is not orthogonal.

4. Since $\begin{bmatrix}2\\-5\\-3\end{bmatrix}\cdot\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}2\\-5\\-3\end{bmatrix}\cdot\begin{bmatrix}4\\-2\\6\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\cdot\begin{bmatrix}4\\-2\\6\end{bmatrix}=0,$ the set is orthogonal.

5. Since $\begin{bmatrix}3\\-2\\1\\3\end{bmatrix}\cdot\begin{bmatrix}-1\\3\\-3\\4\end{bmatrix}=\begin{bmatrix}3\\-2\\1\\3\end{bmatrix}\cdot\begin{bmatrix}3\\8\\7\\0\end{bmatrix}=\begin{bmatrix}-1\\3\\-3\\4\end{bmatrix}\cdot\begin{bmatrix}3\\8\\7\\0\end{bmatrix}=0,$ the set is orthogonal.

6. Since $\begin{bmatrix}-4\\1\\-3\\8\end{bmatrix}\cdot\begin{bmatrix}3\\3\\5\\-1\end{bmatrix}=-32\neq 0,$ the set is not orthogonal.

7. Since $\mathbf{u}_1\cdot\mathbf{u}_2=12-12=0,$ $\{\mathbf{u}_1,\mathbf{u}_2\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent by Theorem 4. Two such vectors in $\mathbb{R}^2$ automatically form a basis for $\mathbb{R}^2$. So $\{\mathbf{u}_1,\mathbf{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$. By Theorem 5,

$$\mathbf{x}=\frac{\mathbf{x}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1+\frac{\mathbf{x}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}=3\mathbf{u}_1+\frac{1}{2}\mathbf{u}_2$$

8. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = -6 + 6 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent by Theorem 4. Two such vectors in $\mathbb{R}^2$ automatically form a basis for $\mathbb{R}^2$. So $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$. By Theorem 5,

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} = -\frac{3}{2}\mathbf{u}_1 + \frac{3}{4}\mathbf{u}_2$$

9. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent by Theorem 4. Three such vectors in $\mathbb{R}^3$ automatically form a basis for $\mathbb{R}^3$. So $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. By Theorem 5,

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} + \frac{\mathbf{x} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = \frac{5}{2}\mathbf{u}_1 - \frac{3}{2}\mathbf{u}_2 + 2\mathbf{u}_3$$

10. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent by Theorem 4. Three such vectors in $\mathbb{R}^3$ automatically form a basis for $\mathbb{R}^3$. So $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. By Theorem 5,

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} + \frac{\mathbf{x} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = \frac{4}{3}\mathbf{u}_1 + \frac{1}{3}\mathbf{u}_2 + \frac{1}{3}\mathbf{u}_3$$

11. Let $\mathbf{y} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. The orthogonal projection of $\mathbf{y}$ onto the line through $\mathbf{u}$ and the origin is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$, and this vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{1}{2}\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

12. Let $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. The orthogonal projection of $\mathbf{y}$ onto the line through $\mathbf{u}$ and the origin is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$, and this vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = -\frac{2}{5}\mathbf{u} = \begin{bmatrix} 2/5 \\ -6/5 \end{bmatrix}$$

13. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$ is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = -\frac{13}{65}\mathbf{u} = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix}$$

The component of $\mathbf{y}$ orthogonal to $\mathbf{u}$ is

$$\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$$

Thus $\mathbf{y} = \hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}}) = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$.

14. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$ is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{2}{5}\mathbf{u} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$$

The component of $\mathbf{y}$ orthogonal to $\mathbf{u}$ is

$$\mathbf{y}-\hat{\mathbf{y}}=\begin{bmatrix}-4/5\\28/5\end{bmatrix}$$

Thus $\mathbf{y}=\hat{\mathbf{y}}+(\mathbf{y}-\hat{\mathbf{y}})=\begin{bmatrix}14/5\\2/5\end{bmatrix}+\begin{bmatrix}-4/5\\28/5\end{bmatrix}$.

15. The distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin is $\|\mathbf{y}-\hat{\mathbf{y}}\|$. One computes that

$$\mathbf{y}-\hat{\mathbf{y}}=\mathbf{y}-\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}=\begin{bmatrix}3\\1\end{bmatrix}-\frac{3}{10}\begin{bmatrix}8\\6\end{bmatrix}=\begin{bmatrix}3/5\\-4/5\end{bmatrix}$$

so $\|\mathbf{y}-\hat{\mathbf{y}}\|=\sqrt{9/25+16/25}=1$ is the desired distance.

16. The distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin is $\|\mathbf{y}-\hat{\mathbf{y}}\|$. One computes that

$$\mathbf{y}-\hat{\mathbf{y}}=\mathbf{y}-\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}=\begin{bmatrix}-3\\9\end{bmatrix}-3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-6\\3\end{bmatrix}$$

so $\|\mathbf{y}-\hat{\mathbf{y}}\|=\sqrt{36+9}=3\sqrt{5}$ is the desired distance.

17. Let $\mathbf{u}=\begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}-1/2\\0\\1/2\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=0$, $\{\mathbf{u},\mathbf{v}\}$ is an orthogonal set. However, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1/3$ and $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=1/2$, so $\{\mathbf{u},\mathbf{v}\}$ is not an orthonormal set. The vectors $\mathbf{u}$ and $\mathbf{v}$ may be normalized to form the orthonormal set

$$\left\{\frac{\mathbf{u}}{\|\mathbf{u}\|},\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\}=\left\{\begin{bmatrix}\sqrt{3}/3\\\sqrt{3}/3\\\sqrt{3}/3\end{bmatrix},\begin{bmatrix}-\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix}\right\}$$

18. Let $\mathbf{u}=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}0\\-1\\0\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=-1\neq0$, $\{\mathbf{u},\mathbf{v}\}$ is not an orthogonal set.

19. Let $\mathbf{u}=\begin{bmatrix}-.6\\.8\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}.8\\.6\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=0$, $\{\mathbf{u},\mathbf{v}\}$ is an orthogonal set. Also, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1$ and $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=1$, so $\{\mathbf{u},\mathbf{v}\}$ is an orthonormal set.

20. Let $\mathbf{u}=\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}1/3\\2/3\\0\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=0$, $\{\mathbf{u},\mathbf{v}\}$ is an orthogonal set. However, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1$ and $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=5/9$, so $\{\mathbf{u},\mathbf{v}\}$ is not an orthonormal set. The vectors $\mathbf{u}$ and $\mathbf{v}$ may be normalized to form the orthonormal set

$$\left\{\frac{\mathbf{u}}{\|\mathbf{u}\|},\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\}=\left\{\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix},\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}\right\}$$

21. Let $\mathbf{u}=\begin{bmatrix}1/\sqrt{10}\\3/\sqrt{20}\\3/\sqrt{20}\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}3/\sqrt{10}\\-1/\sqrt{20}\\-1/\sqrt{20}\end{bmatrix}$, and $\mathbf{w}=\begin{bmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}=\mathbf{v}\cdot\mathbf{w}=0$, $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is an orthogonal set. Also, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1$, $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=1$, and $\|\mathbf{w}\|^2=\mathbf{w}\cdot\mathbf{w}=1$, so $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is an orthonormal set.

22. Let $\mathbf{u}=\begin{bmatrix}1/\sqrt{18}\\4/\sqrt{18}\\1/\sqrt{18}\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}$, and $\mathbf{w}=\begin{bmatrix}-2/3\\1/3\\-2/3\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}=\mathbf{v}\cdot\mathbf{w}=0$, $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is an orthogonal set. Also, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1$, $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=1$, and $\|\mathbf{w}\|^2=\mathbf{w}\cdot\mathbf{w}=1$, so $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is an orthonormal set.

23. **a.** True. For example, the vectors **u** and **y** in Example 3 are linearly independent but not orthogonal.
 b. True. The formulas for the weights are given in Theorem 5.
 c. False. See the paragraph following Example 5.
 d. False. The matrix must also be square. See the paragraph before Example 7.
 e. False. See Example 4. The distance is $\|\mathbf{y}-\hat{\mathbf{y}}\|$.

24. **a.** True. But every orthogonal set of *nonzero vectors* is linearly independent. See Theorem 4.
 b. False. To be orthonormal, the vectors is S must be unit vectors as well as being orthogonal to each other.
 c. True. See Theorem 7(a).
 d. True. See the paragraph before Example 3.
 e. True. See the paragraph before Example 7.

25. To prove part (b), note that

$$(U\mathbf{x})\cdot(U\mathbf{y})=(U\mathbf{x})^T(U\mathbf{y})=\mathbf{x}^TU^TU\mathbf{y}=\mathbf{x}^T\mathbf{y}=\mathbf{x}\cdot\mathbf{y}$$

because $U^TU=I$. If $\mathbf{y}=\mathbf{x}$ in part (b), $(U\mathbf{x})\cdot(U\mathbf{x})=\mathbf{x}\cdot\mathbf{x}$, which implies part (a). Part (c) of the Theorem follows immediately fom part (b).

26. A set of n nonzero orthogonal vectors must be linearly independent by Theorem 4, so if such a set spans W it is a basis for W. Thus W is an n-dimensional subspace of $\mathbb{R}^n$, and $W=\mathbb{R}^n$.

27. If U has orthonormal columns, then $U^TU=I$ by Theorem 6. If U is also a square matrix, then the equation $U^TU=I$ implies that U is invertible by the Invertible Matrix Theorem.

28. If U is an $n\times n$ orthogonal matrix, then $I=UU^{-1}=UU^T$. Since U is the transpose of U^T, Theorem 6 applied to U^T says that U^T has orthogonal columns. In particular, the columns of U^T are linearly independent and hence form a basis for $\mathbb{R}^n$ by the Invertible Matrix Theorem. That is, the rows of U form a basis (an orthonormal basis) for $\mathbb{R}^n$.

29. Since U and V are orthogonal, each is invertible. By Theorem 6 in Section 2.2, UV is invertible and $(UV)^{-1}=V^{-1}U^{-1}=V^TU^T=(UV)^T$, where the final equality holds by Theorem 3 in Section 2.1. Thus UV is an orthogonal matrix.

30. If U is an orthogonal matrix, its columns are orthonormal. Interchanging the columns does not change their orthonormality, so the new matrix – say, V – still has orthonormal columns. By Theorem 6, $V^TV = I$. Since V is square, $V^T = V^{-1}$ by the Invertible Matrix Theorem.

31. Suppose that $\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$. Replacing $\mathbf{u}$ by $c\mathbf{u}$ with $c \neq 0$ gives

$$\frac{\mathbf{y}\cdot(c\mathbf{u})}{(c\mathbf{u})\cdot(c\mathbf{u})}(c\mathbf{u}) = \frac{c(\mathbf{y}\cdot\mathbf{u})}{c^2(\mathbf{u}\cdot\mathbf{u})}(c)\mathbf{u} = \frac{c^2(\mathbf{y}\cdot\mathbf{u})}{c^2(\mathbf{u}\cdot\mathbf{u})}\mathbf{u} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \hat{\mathbf{y}}$$

So $\hat{\mathbf{y}}$ does not depend on the choice of a nonzero $\mathbf{u}$ in the line L used in the formula.

32. If $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, then by Theorem 1(c) in Section 6.1,

$$(c_1\mathbf{v}_1)\cdot(c_2\mathbf{v}_2) = c_1[\mathbf{v}_1\cdot(c_2\mathbf{v}_2)] = c_1c_2(\mathbf{v}_1\cdot\mathbf{v}_2) = c_1c_2 0 = 0$$

33. Let $L = \text{Span}\{\mathbf{u}\}$, where $\mathbf{u}$ is nonzero, and let $T(\mathbf{x}) = \frac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$. For any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and any scalars c and d, the properties of the inner product (Theorem 1) show that

$$\begin{aligned} T(c\mathbf{x}+d\mathbf{y}) &= \frac{(c\mathbf{x}+d\mathbf{y})\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \\ &= \frac{c\mathbf{x}\cdot\mathbf{u}+d\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \\ &= \frac{c\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} + \frac{d\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \\ &= cT(\mathbf{x}) + dT(\mathbf{y}) \end{aligned}$$

Thus T is a linear transformation. Another approach is to view T as the composition of the following three linear mappings: $\mathbf{x} \mapsto a = \mathbf{x}\cdot\mathbf{v}$, $a \mapsto b = a / \mathbf{v}\cdot\mathbf{v}$, and $b \mapsto b\mathbf{v}$.

34. Let $L = \text{Span}\{\mathbf{u}\}$, where $\mathbf{u}$ is nonzero, and let $T(\mathbf{x}) = \text{refl}_L\mathbf{y} = 2\text{proj}_L\mathbf{y} - \mathbf{y}$. By Exercise 33, the mapping $\mathbf{y} \mapsto \text{proj}_L\mathbf{y}$ is linear. Thus for any vectors $\mathbf{y}$ and $\mathbf{z}$ in $\mathbb{R}^n$ and any scalars c and d,

$$\begin{aligned} T(c\mathbf{y}+d\mathbf{z}) &= 2\,\text{proj}_L(c\mathbf{y}+d\mathbf{z}) - (c\mathbf{y}+d\mathbf{z}) \\ &= 2(c\,\text{proj}_L\mathbf{y} + d\,\text{proj}_L\mathbf{z}) - c\mathbf{y} - d\mathbf{z} \\ &= 2c\,\text{proj}_L\mathbf{y} - c\mathbf{y} + 2d\,\text{proj}_L\mathbf{z} - d\mathbf{z} \\ &= c(2\,\text{proj}_L\mathbf{y} - \mathbf{y}) + d(2\,\text{proj}_L\mathbf{z} - \mathbf{z}) \\ &= cT(\mathbf{y}) + dT(\mathbf{z}) \end{aligned}$$

Thus T is a linear transformation.

35. **[M]** One can compute that $A^TA = 100I_4$. Since the off-diagonal entries in A^TA are zero, the columns of A are orthogonal.

36. [M]

a. One computes that $U^T U = I_4$, while

$$UU^T = \left(\frac{1}{100}\right)\begin{bmatrix} 82 & 0 & -20 & 8 & 6 & 20 & 24 & 0 \\ 0 & 42 & 24 & 0 & -20 & 6 & 20 & -32 \\ -20 & 24 & 58 & 20 & 0 & 32 & 0 & 6 \\ 8 & 0 & 20 & 82 & 24 & -20 & 6 & 0 \\ 6 & -20 & 0 & 24 & 18 & 0 & -8 & 20 \\ 20 & 6 & 32 & -20 & 0 & 58 & 0 & 24 \\ 24 & 20 & 0 & 6 & -8 & 0 & 18 & -20 \\ 0 & -32 & 6 & 0 & 20 & 24 & -20 & 42 \end{bmatrix}$$

The matrices $U^T U$ and UU^T are of different sizes and look nothing like each other.

b. Answers will vary. The vector $\mathbf{p} = UU^T\mathbf{y}$ is in Col U because $\mathbf{p} = U(U^T\mathbf{y})$. Since the columns of U are simply scaled versions of the columns of A, Col U = Col$A$. Thus each $\mathbf{p}$ is in Col A.

c. One computes that $U^T\mathbf{z} = \mathbf{0}$.

d. From (c), **z** is orthogonal to each column of A. By Exercise 29 in Section 6.1, **z** must be orthogonal to every vector in Col A; that is, **z** is in $(\text{Col } A)^\perp$.