6.3 SOLUTIONS

Notes: Example 1 seems to help students understand Theorem 8. Theorem 8 is needed for the Gram-Schmidt process (but only for a subspace that itself has an orthogonal basis). Theorems 8 and 9 are needed for the discussions of least squares in Sections 6.5 and 6.6. Theorem 10 is used with the QR factorization to provide a good numerical method for solving least squares problems, in Section 6.5. Exercises 19 and 20 lead naturally into consideration of the Gram-Schmidt process.

1. The vector in $\text{Span}\{\mathbf{u}_4\}$ is

$$\frac{\mathbf{x}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\mathbf{u}_4 = \frac{72}{36}\mathbf{u}_4 = 2\mathbf{u}_4 = \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$$

Since $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + \frac{\mathbf{x}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\mathbf{u}_4$, the vector

$$\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\mathbf{u}_4 = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix}$$

is in $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

2. The vector in $\text{Span}\{\mathbf{u}_1\}$ is

$$\frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 = \frac{14}{7}\mathbf{u}_1 = 2\mathbf{u}_1 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

Since $\mathbf{x} = \frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$, the vector

$$\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix}$$

is in $\text{Span}\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

3. Since $\mathbf{u}_1\cdot\mathbf{u}_2 = -1+1+0 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. The orthogonal projection of **y** onto $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \frac{3}{2}\mathbf{u}_1 + \frac{5}{2}\mathbf{u}_2 = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

4. Since $\mathbf{u}_1\cdot\mathbf{u}_2 = -12+12+0 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. The orthogonal projection of **y** onto $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \frac{30}{25}\mathbf{u}_1 - \frac{15}{25}\mathbf{u}_2 = \frac{6}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - \frac{3}{5}\begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

5. Since $\mathbf{u}_1\cdot\mathbf{u}_2 = 3+1-4 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. The orthogonal projection of **y** onto $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \frac{7}{14}\mathbf{u}_1 - \frac{15}{6}\mathbf{u}_2 = \frac{1}{2}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$$

6. Since $\mathbf{u}_1\cdot\mathbf{u}_2 = 0-1+1 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. The orthogonal projection of **y** onto $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = -\frac{27}{18}\mathbf{u}_1 + \frac{5}{2}\mathbf{u}_2 = -\frac{3}{2}\begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

7. Since $\mathbf{u}_1\cdot\mathbf{u}_2 = 5+3-8 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. By the Orthogonal Decomposition Theorem,

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = 0\mathbf{u}_1 + \frac{2}{3}\mathbf{u}_2 = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix}, \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}$$

and $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

8. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = -1+3-2=0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. By the Orthogonal Decomposition Theorem,

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = 2\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}, \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$$

and $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

9. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. By the Orthogonal Decomposition Theorem,

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3 = 2\mathbf{u}_1 + \frac{2}{3}\mathbf{u}_2 - \frac{2}{3}\mathbf{u}_3 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

and $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

10. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. By the Orthogonal Decomposition Theorem,

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3 = \frac{1}{3}\mathbf{u}_1 + \frac{14}{3}\mathbf{u}_2 - \frac{5}{3}\mathbf{u}_3 = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}, \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

and $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

11. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. The Best Approximation Theorem says that $\hat{\mathbf{y}}$, which is the orthogonal projection of $\mathbf{y}$ onto $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, is the closest point to $\mathbf{y}$ in W. This vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1 + \frac{3}{2}\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

12. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. The Best Approximation Theorem says that $\hat{\mathbf{y}}$, which is the orthogonal projection of y onto $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, is the closest point to $\mathbf{y}$ in W. This vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = 3\mathbf{v}_1 + 1\mathbf{v}_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$$

13. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. By the Best Approximation Theorem, the closest point in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\mathbf{z}$ is

$$\hat{\mathbf{z}} = \frac{\mathbf{z}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{z}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = \frac{2}{3}\mathbf{v}_1 - \frac{7}{3}\mathbf{v}_2 = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 3 \end{bmatrix}$$

14. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. By the Best Approximation Theorem, the closest point in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\mathbf{z}$ is

$$\hat{\mathbf{z}} = \frac{\mathbf{z}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{z}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1 + 0\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ -3/2 \end{bmatrix}$$

15. The distance from the point $\mathbf{y}$ in $\mathbb{R}^3$ to a subspace W is defined as the distance from $\mathbf{y}$ to the closest point in W. Since the closest point in W to $\mathbf{y}$ is $\hat{\mathbf{y}} = \text{proj}_W\mathbf{y}$, the desired distance is $\|\mathbf{y} - \hat{\mathbf{y}}\|$. One computes that $\hat{\mathbf{y}} = \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}, \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$, and $\|\mathbf{y} - \hat{\mathbf{y}}\| = \sqrt{40} = 2\sqrt{10}$.

16. The distance from the point $\mathbf{y}$ in $\mathbb{R}^4$ to a subspace W is defined as the distance from $\mathbf{y}$ to the closest point in W. Since the closest point in W to $\mathbf{y}$ is $\hat{\mathbf{y}} = \text{proj}_W\mathbf{y}$, the desired distance is $\|\mathbf{y} - \hat{\mathbf{y}}\|$. One computes that $\hat{\mathbf{y}} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}, \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$, and $\|\mathbf{y} - \hat{\mathbf{y}}\| = 8$.

17. **a.** $U^TU = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, UU^T = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix}$

b. Since $U^TU = I_2$, the columns of U form an orthonormal basis for W, and by Theorem 10

$$\text{proj}_W\mathbf{y} = UU^T\mathbf{y} = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}.$$

18. **a.** $U^TU = [1] = 1, UU^T = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}$

b. Since $U^TU = 1$, $\{\mathbf{u}_1\}$ forms an orthonormal basis for W, and by Theorem 10

$$\text{proj}_W\mathbf{y} = UU^T\mathbf{y} = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}\begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

19. By the Orthogonal Decomposition Theorem, $\mathbf{u}_3$ is the sum of a vector in $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and a vector $\mathbf{v}$ orthogonal to W. This exercise asks for the vector $\mathbf{v}$:

$$\mathbf{v} = \mathbf{u}_3 - \text{proj}_W\mathbf{u}_3 = \mathbf{u}_3 - \left(-\frac{1}{3}\mathbf{u}_1 + \frac{1}{15}\mathbf{u}_2\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -2/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \\ 1/5 \end{bmatrix}$$

Any multiple of the vector $\mathbf{v}$ will also be in $W^\perp$.

20. By the Orthogonal Decomposition Theorem, $\mathbf{u}_4$ is the sum of a vector in $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and a vector $\mathbf{v}$ orthogonal to W. This exercise asks for the vector $\mathbf{v}$:

$$\mathbf{v} = \mathbf{u}_4 - \text{proj}_W \mathbf{u}_4 = \mathbf{u}_4 - \left(\frac{1}{6}\mathbf{u}_1 - \frac{1}{30}\mathbf{u}_2\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/5 \\ -2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4/5 \\ 2/5 \end{bmatrix}$$

Any multiple of the vector $\mathbf{v}$ will also be in $W^\perp$.

21. **a.** True. See the calculations for $\mathbf{z}_2$ in Example 1 or the box after Example 6 in Section 6.1.

b. True. See the Orthogonal Decomposition Theorem.

c. False. See the last paragraph in the proof of Theorem 8, or see the second paragraph after the statement of Theorem 9.

d. True. See the box before the Best Approximation Theorem.

e. True. Theorem 10 applies to the column space W of U because the columns of U are linearly independent and hence form a basis for W.

22. **a.** True. See the proof of the Orthogonal Decomposition Theorem.

b. True. See the subsection "A Geometric Interpretation of the Orthogonal Projection."

c. True. The orthgonal decomposition in Theorem 8 is unique.

d. False. The Best Approximation Theorem says that the best approximation to $\mathbf{y}$ is $\text{proj}_W \mathbf{y}$.

e. False. This statement is only true if $\mathbf{x}$ is in the column space of U. If $n > p$, then the column space of U will not be all of $\mathbb{R}^n$, so the statement cannot be true for all $\mathbf{x}$ in $\mathbb{R}^n$.

23. By the Orthogonal Decomposition Theorem, each $\mathbf{x}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{x} = \mathbf{p} + \mathbf{u}$, with $\mathbf{p}$ in Row A and $\mathbf{u}$ in $(\text{Row } A)^\perp$. By Theorem 3 in Section 6.1, $(\text{Row } A)^\perp = \text{Nul}\, A$, so $\mathbf{u}$ is in Nul$A$.

Next, suppose $A\mathbf{x} = \mathbf{b}$ is consistent. Let $\mathbf{x}$ be a solution and write $\mathbf{x} = \mathbf{p} + \mathbf{u}$ as above. Then $A\mathbf{p} = A(\mathbf{x} - \mathbf{u}) = A\mathbf{x} - A\mathbf{u} = \mathbf{b} - \mathbf{0} = \mathbf{b}$, so the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution $\mathbf{p}$ in Row A.

Finally, suppose that $\mathbf{p}$ and $\mathbf{p}_1$ are both in Row$A$ and both satisfy $A\mathbf{x} = \mathbf{b}$. Then $\mathbf{p} - \mathbf{p}_1$ is in $\text{Nul}\, A = (\text{Row } A)^\perp$, since $A(\mathbf{p} - \mathbf{p}_1) = A\mathbf{p} - A\mathbf{p}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0}$. The equations $\mathbf{p} = \mathbf{p}_1 + (\mathbf{p} - \mathbf{p}_1)$ and $\mathbf{p} = \mathbf{p} + \mathbf{0}$ both then decompose $\mathbf{p}$ as the sum of a vector in Row$A$ and a vector in $(\text{Row } A)^\perp$. By the uniqueness of the orthogonal decomposition (Theorem 8), $\mathbf{p} = \mathbf{p}_1$, and $\mathbf{p}$ is unique.

24. **a.** By hypothesis, the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_p$ are pairwise orthogonal, and the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_q$ are pairwise orthogonal. Since $\mathbf{w}_i$ is in W for any i and $\mathbf{v}_j$ is in $W^\perp$ for any j, $\mathbf{w}_i \cdot \mathbf{v}_j = 0$ for any i and j. Thus $\{\mathbf{w}_1, \ldots, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ forms an orthogonal set.

b. For any $\mathbf{y}$ in $\mathbb{R}^n$, write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ as in the Orthogonal Decomposition Theorem, with $\hat{\mathbf{y}}$ in W and $\mathbf{z}$ in $W^\perp$. Then there exist scalars $c_1, \ldots, c_p$ and $d_1, \ldots, d_q$ such that $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = c_1\mathbf{w}_1 + \ldots + c_p\mathbf{w}_p + d_1\mathbf{v}_1 + \ldots + d_q\mathbf{v}_q$. Thus the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ spans $\mathbb{R}^n$.

c. The set $\{\mathbf{w}_1, \ldots, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ is linearly independent by (a) and spans $\mathbb{R}^n$ by (b), and is thus a basis for $\mathbb{R}^n$. Hence $\dim W + \dim W^\perp = p + q = \dim \mathbb{R}^n$.

25. **[M]** Since $U^TU = I_4$, U has orthonormal columns by Theorem 6 in Section 6.2. The closest point to **y** in Col U is the orthogonal projection $\hat{\mathbf{y}}$ of **y** onto Col U. From Theorem 10,

$$\hat{\mathbf{y}} = UU^{\mathrm{T}}\mathbf{y} = \begin{bmatrix} 1.2 \\ .4 \\ 1.2 \\ 1.2 \\ .4 \\ 1.2 \\ .4 \\ .4 \end{bmatrix}$$

26. **[M]** The distance from **b** to Col U is $\|\mathbf{b} - \hat{\mathbf{b}}\|$, where $\hat{\mathbf{b}} = UU^{\mathrm{T}}\mathbf{b}$. One computes that

$$\hat{\mathbf{b}} = UU^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} .2 \\ .92 \\ .44 \\ 1 \\ -.2 \\ -.44 \\ .6 \\ -.92 \end{bmatrix}, \mathbf{b} - \hat{\mathbf{b}} = \begin{bmatrix} .8 \\ .08 \\ .56 \\ 0 \\ -.8 \\ -.56 \\ -1.6 \\ -.08 \end{bmatrix}, \|\mathbf{b} - \hat{\mathbf{b}}\| = \frac{\sqrt{112}}{5}$$

which is 2.1166 to four decimal places.