1.3 Vector Equations
**Vector Equations.**

Vectors in $\mathbb{R}^2$.

A column vector in $\mathbb{R}^2$ is of the form $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ where $x$ and $y$ are any real numbers. The same vector can also be written as a row vector $\vec{v} = \begin{bmatrix} x & y \end{bmatrix}$.

Two vectors in $\mathbb{R}^2$ are equal if and only if corresponding entries are equal: if $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} r \\ s \end{bmatrix}$, then $\vec{v} = \vec{w} \iff x = r$ and $y = s$. 
**Vector Arithmetic.**

Let vectors \( \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \), \( \vec{w} = \begin{bmatrix} r \\ s \end{bmatrix} \) \( \in \mathbb{R}^2 \) and let \( c \in \mathbb{R} \).

Then \( \vec{v} \pm \vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \pm \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} x \pm r \\ y \pm s \end{bmatrix} \) and \( c \cdot \vec{v} = c \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} \).

These are the definitions of *vector addition* and *scalar multiplication*.

Geometrically, a vector \( \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \) can be thought of as an *ordered pair* \((x, y)\) in the plane; \( \mathbb{R}^2 \) is the set of all points in the plane.
Vectors in $\mathbb{R}^3$ and $\mathbb{R}^n$.

A column vector in $\mathbb{R}^3$ is of the form $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x, y$ and $z$ are any real numbers.

A column vector in $\mathbb{R}^n$ is of the form $\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}$ where again, the $u_j$ are any real numbers for all $j, j = 1, \ldots, n$. 
Algebraic Properties of $\mathbb{R}^n$.

For all vectors $u, v, w \in \mathbb{R}^n$ and scalars $c, d \in \mathbb{R}$,

(i) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
(ii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
(iii) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$
(iv) $\vec{u} + (-\vec{u}) = -\vec{u} + \vec{u} = \vec{0}$

(v) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
(vi) $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
(vii) $c(d\vec{u}) = (cd)\vec{u}$
(viii) $1\vec{u} = \vec{u}$

and $-\vec{u} = (\vec{u})$
**Linear combinations.**

Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_p \in \mathbb{R}$, the vector

$$\vec{y} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_p \vec{v}_p$$

is called a **linear combination** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ with weights $c_1, c_2, \ldots, c_p$. 
A vector equation

\[ c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n = \vec{b} \]

has the same solution set as the linear system whose augmented matrix is

\[
\begin{bmatrix}
\vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n & \vec{b}
\end{bmatrix}.
\]

\( \vec{b} \) can be generated by a linear combination of \( c_1, c_2, \ldots, c_n \) if and only if there exists a solution to the linear system corresponding to

\[
\begin{bmatrix}
\vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n & \vec{b}
\end{bmatrix}.
\]
Definition.

If \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n \), then the set of all linear combinations of \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \) is denoted by \( \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \) and is called the subset of \( \mathbb{R}^n \) spanned by \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \).

Thus \( \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \) is the collection of all vectors that can be written as a linear combination

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_p \vec{v}_p
\]

with \( c_1, c_2, \ldots, c_p \) scalars.
For example, \( \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p\} \) contains every scalar multiple of \( \vec{v}_1 \) since 
\[
c \vec{v}_1 = c_1 \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_p.
\]

Also, the zero vector, \( \vec{0} \), must be in \( \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p\} \).
For example,

\[
\begin{align*}
\vec{v}_1 &= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \\
\vec{v}_2 &= \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \\
\vec{v}_3 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.
\end{align*}
\]

Does \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \) span \( \mathbb{R}^4 \)?

Why or why not?
Solution: Row reduce the matrix \([\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]\) to determine whether it has a pivot in each row.

\[
\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The matrix \([\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]\) does not have a pivot in each row, so the columns of the matrix do not span \(\mathbb{R}^4\).

That is, \(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}\) does not span \(\mathbb{R}^4\).
Question: Could a set of three vectors in \( \mathbb{R}^4 \) span all of \( \mathbb{R}^4 \)?

What about \( n \) vectors in \( \mathbb{R}^m \) when \( n \) is less than \( m \)?
A set of three vectors in $\mathbb{R}^4$ cannot span $\mathbb{R}^4$.

Reason:
The matrix $A$ whose columns are these three vectors has four rows. To have a pivot in each row, $A$ would have to have at least four columns - one for each pivot.

That is not the case here. Since $A$ does not have a pivot in every row, its columns do not span $\mathbb{R}^4$.

In general, a set of $n$ vectors in $\mathbb{R}^m$ cannot span $\mathbb{R}^m$ when $n$ is less than $m$. 
End Presentation