6.2 Orthogonal Sets
Definition:

A set of vectors \{\vec{u}_1,\ldots,\vec{u}_p\} in \mathbb{R}^n is said to be orthogonal if each pair of distinct vectors from the set is orthogonal, which means \( \vec{u}_i \cdot \vec{u}_j = 0 \) whenever \( i \neq j \).

Theorem:

If \( S = \{\vec{u}_1,\ldots,\vec{u}_p\} \) is an orthogonal set of nonzero vectors in \( \mathbb{R}^n \), then \( S \) is linearly independent and thus is a basis for the subspace spanned by \( S \).
Definition:

An orthogonal basis for a subspace $W$ is also an orthogonal set.

Theorem:

Let $\{\vec{u}_1, \ldots, \vec{u}_p\}$ be an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$. For each $\vec{y}$ in $W$, the weights in the linear combination

$$\vec{y} = c_1\vec{u}_1 + \cdots + c_p\vec{u}_p$$

are given by

$$c_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j} \quad j = 1, \ldots, p$$
Orthogonal Projections:

Given a nonzero vector \( \vec{u} \) in \( \mathbb{R}^n \), we want to decompose a vector \( \vec{y} \) into the sum of two vectors, one a multiple of \( \vec{u} \) and the other orthogonal to \( \vec{u} \).

We want \( \vec{y} = \hat{\vec{y}} + \vec{z} \) where \( \hat{\vec{y}} = \alpha \vec{u} \) for some scalar \( \alpha \) and \( \vec{z} \) is some vector orthogonal to \( \vec{u} \).

Given any scalar \( \alpha \), let \( \vec{z} = \vec{y} - \alpha \vec{u} \). Then \( \vec{y} - \hat{\vec{y}} \) is orthogonal to \( \vec{u} \) if and only if

\[
0 = (\vec{y} - \alpha \vec{u}) \cdot \vec{u} = \vec{y} \cdot \vec{u} - (\alpha \vec{u}) \cdot \vec{u} = \vec{y} \cdot \vec{u} - \alpha (\vec{u} \cdot \vec{u})
\]
We have \( \tilde{y} = \hat{y} + \bar{z} \) satisfied with \( \bar{z} \) orthogonal to \( \bar{u} \) if and only if

\[
\alpha = \frac{\tilde{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \quad \text{and} \quad \hat{y} = \frac{\tilde{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \bar{u}
\]

The vector \( \hat{y} \) is called the *orthogonal projection of \( \tilde{y} \) onto \( \bar{u} \).*

The vector \( \bar{z} \) is called the *component of \( \tilde{y} \) orthogonal to \( \bar{u} \).*
If \( c \) is any scalar and if \( \vec{u} \) is replaced by \( c\vec{u} \) in the definition of \( \hat{\vec{y}} \), then the orthogonal projection of \( \vec{y} \) onto \( c\vec{u} \) is exactly the same as the orthogonal projection of \( \vec{y} \) onto \( \vec{u} \).

Thus this projection is determined by the subspace \( L \) spanned by \( \vec{u} \).

\[
\hat{\vec{y}} = \text{proj}_{L} \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}
\]
Orthonormal Sets:

A set \( \{\vec{u}_1, \ldots, \vec{u}_p\} \) is an orthonormal set if it is an orthogonal set of unit vectors.

If \( W \) is the subspace spanned by such a set, then \( \{\vec{u}_1, \ldots, \vec{u}_p\} \) is an orthonormal basis for \( W \), since the set is automatically linearly independent. (Theorem 4).

The standard basis \( \{\vec{e}_1, \ldots, \vec{e}_p\} \) for \( \mathbb{R}^n \) is an orthonormal set.
Theorem: An $m \times n$ matrix $U$ has orthonormal columns if and only if

$$U^T U = I.$$ 

Theorem: Let $U$ be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0 \iff \mathbf{x} \cdot \mathbf{y} = 0$. 

End presentation