Review topics. Integration techniques ...

This is NOT a course in integration, however, you should be able to handle the following at a minimum:

1. Integration by Substitution

2. Integration by Parts

3. Partial Fraction Decomposition
Partial Fraction Decomposition. Evaluate \( I = \int \frac{x+1}{(x-1)(x-3)} \, dx \)

We try to use \( \frac{x+1}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3} \) to determine constants \( A \) and \( B \). To determine them, note that

\[
\frac{A}{x-1} + \frac{B}{x-3} = \frac{(A+B)x-3A-B}{(x-1)(x-3)} = \frac{x+1}{(x-1)(x-3)}
\]

Equating like powers of \( x \), we have \( A+B = 1 \) and \( -3A - B = 1 \). Solving, we get \( A = -1 \) and \( B = 2 \).

Thus, \( \frac{x+1}{(x-1)(x-3)} = -\frac{1}{x-1} + \frac{2}{x-3} \).
So 
\[ I = \int \frac{x+1}{(x-1)(x-3)} \, dx = -\int \frac{dx}{x-1} + \int \frac{2\,dx}{x-3}. \]

and 
\[ I = -\ln |x-1| + 2 \ln |x-3| + C. \]

or simplifying, 
\[ I = \ln \left( \frac{|x-3|^2}{|x-1|} \right) + C. \]
The topic of study is the ordinary differential equation - an equation containing the derivatives of an unknown function $x(t)$, with $t$ a real variable, possibly containing the unknown function itself, and the independent variable $t$.

In addition, initial conditions, which the unknown function is required to satisfy, may also be given.
Given such an equation, the object is two-fold:

(i) To find the unknown function or class of functions satisfying the equation, and

(ii) Whether or not (i) is possible, to gain some information about the behavior of any function satisfying the equation.
DEFINITION: The order of a differential equation is the order of the highest derivative of the unknown function appearing in the equation.

So the general form of an ordinary differential equation of $k$th order is

$$F\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^k x}{dt^k}\right) = 0,$$

where $x = x(t)$ is an unknown function and $F$ is defined on some domain $B$. 
EXAMPLE:

A second order equation \( a(t) \frac{d^2x}{dt^2} + b(t) \frac{dx}{dt} + c(t) x = 0 \)

where \( x = x(t) \) is an unknown function and the coefficients \( a(t), b(t), \) and \( c(t) \) are continuous functions, with \( a(t) \neq 0 \), defined on some interval \( t_1 < t < t_2 \).
DEFINITION: A function $x = \varphi(t)$, $t_1 < t < t_2$, which when substituted into $F\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^k x}{dt^k}\right) = 0$ reduces it to an identity, is called a solution of the equation, and $(t_1, t_2)$ is its interval of definition.
Suppose we are given an equation of the form

\[ a \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + cx = 0 \]

where \( x = x(t) \) is an unknown function and the coefficients \( a, b, \) and \( c \) are constant functions defined on some interval \( t_1 < t < t_2, \) and \( a \neq 0 \)

We can reduce this equation to a system of first-order differential equations ....
To begin, we simplify \[ a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx = 0 \]

by setting \( p = \frac{b}{a} \), and \( q = \frac{c}{a} \).

To get

\[ \frac{d^2x}{dt^2} + p \frac{dx}{dt} + qx = 0 \]

We now convert this second-order equation to a pair of first-order equations. We will need another variable to make this work.
We introduce a new variable, \( v = \frac{dx}{dt} \Rightarrow \frac{dv}{dt} = \frac{d^2 x}{dt^2} \).

Then solve for \( \frac{d^2 x}{dt^2} \) in \( \frac{d^2 x}{dt^2} + p \frac{dx}{dt} + qx = 0 \)

and \( \frac{d^2 x}{dt^2} + p \frac{dx}{dt} + qx = 0 \Rightarrow \frac{d^2 x}{dt^2} = -p \frac{dx}{dt} - qx \)

Substituting we get \( \frac{dv}{dt} = -pv - qx \), and we have the

\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= -pv - qx
\end{align*}
\]

first order system
We can write the system as a matrix equation ...

\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= -pv - qx
\end{align*}
\Rightarrow
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dv}{dt}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
-q & -p
\end{bmatrix}
\begin{bmatrix}
x(t) \\
v(t)
\end{bmatrix}
\]
It follows that we need only consider the first-order equation

\[
\frac{dx}{dt} = f(t, x)
\]

\text{(\ast)}

where \( x = x(t) \) is an unknown function and \( f(t, x) \) is defined in some domain \( B \).

If \( f(t, x) \) is continuous in \( B \), and \( x = \varphi(t), t_1 < t < t_2 \), is a solution of \((\ast)\), then it may be thought of as a curve lying entirely in \( B \). Further it will have a continuously turning tangent at each point given by \( \varphi'(t) = f(t, \varphi(t)) \).

Such a curve is often called an \textit{integral curve} or \textit{trajectory}.
Given a point \((t_0, x_0)\) in \(B\), we can compute the value of the function \(f(t_0, x_0)\). If we construct a line segment \(\varphi_{t,x}\) passing through \((t_0, x_0)\) and parallel to \(f(t, x)\), and do this for all \((t, x)\) in \(B\), we get the

**direction field** of \(\frac{dx}{dt} = f(t, x)\).

It follows that any integral curve \(\varphi(t), \; t_1 < t < t_2\) of \(\frac{dx}{dt} = f(t, x)\) is then tangent to \(\varphi_{t,\varphi(t)}\) at each point \((t, \varphi(t))\) of \(B\).
\[
dy/dt = y - t
\]
End presentation