Repeated Eigenvalues - Example
Recall the theorem:

Suppose \( \frac{d\vec{Y}}{dt} = A\vec{Y} \) is a linear system in which the \( 2 \times 2 \) matrix \( A \) has a repeated eigenvalue \( \lambda \) but only one line of eigenvectors. Then the general solution has the form

\[
\vec{Y}(t) = e^{\lambda t} \vec{V}_0 + t e^{\lambda t} \vec{V}_1
\]

where \( \vec{V}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \) is an arbitrary initial condition and \( \vec{V}_1 \) is determined from \( \vec{V}_0 \) by \( \vec{V}_1 = (A - \lambda I)\vec{V}_0 \).

If \( \vec{V}_1 \) is zero, then \( \vec{V}_0 \) is an eigenvector and \( \vec{Y}(t) \) is a straight-line solution. Otherwise, \( \vec{V}_1 \) is an eigenvector.
Find the general solution for the linear system. Then find the particular solution satisfying the given initial condition.

\[
\frac{d\vec{Y}}{dt} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \vec{Y}, \quad \vec{Y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

The characteristic equation is \((2 - \lambda)(4 - \lambda) + 1 = (\lambda - 3)^2 = 0\), and the repeated eigenvalue is \(\lambda = 3\).

To find an associated eigenvector, \(\vec{V} = \begin{bmatrix} x \\ y \end{bmatrix}\), we solve the simultaneous equations

\[
\begin{align*}
2x + y &= 3x \\
-x + 4y &= 3y
\end{align*}
\]

which come from \(A\vec{V} = \lambda\vec{V}\).
\[ 2x + y = 3x \]
\[ -x + 4y = 3y \]
\[ \Rightarrow y = x \] and so representative eigenvector is

\[ \vec{V} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]
Since the eigenvalue is positive, all solutions except the equilibrium solution are unbounded as $t \to \infty$.

As $t \to -\infty$, the solutions with initial conditions that lie in the half-plane $y > x$ approach the origin tangent to the half-line $y = x$ with $y < 0$.

And the solutions with initial conditions that lie in the half-plane $y < x$ approach the origin tangent to the half-line $y = x$ with $y > 0$. 
To find the general solution, we start with the arbitrary initial condition \( \vec{V}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \)

Then

\[
\vec{V}_1 = \left( \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{V}_0 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} y_0 - x_0 \\ y_0 - x_0 \end{bmatrix}
\]

and we obtain the general solution

\[
\vec{Y}(t) = e^{3t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^{3t} \begin{bmatrix} y_0 - x_0 \\ y_0 - x_0 \end{bmatrix}
\]
The solution which satisfies the initial condition \[
\begin{bmatrix}
x_0 \\
y_0
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]
is
\[
\ddot{Y}(t) = e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + te^{3t} \begin{bmatrix} -1 \\ -1 \end{bmatrix}
\]
END