Sec 8.1 The Discrete Logistic equation
Discrete Dynamical Systems

We now move on to a new type of model for time dependent processes.

Discrete dynamical systems generally involve iteration to move from one time step to the next time step.

Thus we have systems which do not vary continuously.

The equations we will encounter are called *difference equations*. 
As with differential equations, finding exact solutions will rarely be possible.

As with nonlinear differential equations, nonlinear difference equations will be difficult to solve completely.

We will need numerical, analytic, and qualitative methods.

We begin by going to the logistic equation in discrete form.
The exponential growth model becomes \( P_{n+1} = kP_n \)

where \( k \) is the growth rate constant and \( P_{n+1} \) represents the actual population at the \((n+1)\)st time step.

Contrast the logistic differential equation \( \frac{dP}{dt} = kP \)

where \( k \) is the growth rate and \( \frac{dP}{dt} \) represents the rate of change in population.
\[ P_{n+1} = kP_n \]

is a kind of initial value problem. If we know the value of \( P_n \) when \( n = 0 \), we can generate a sequence of \( P \) values:

\[
\begin{align*}
P_1 &= kP_0 \\
P_2 &= kP_1 = k(kP_0) = k^2P_0 \\
P_3 &= kP_2 = k(k^2P_0) = k^3P_0 \\
&\vdots \\
P_n &= k^nP_0
\end{align*}
\]
Given \( P_{n+1} = kP_n \) and initial value \( P_0 \)

If \( P_0 > 0 \) then for

\[
k > 1, \quad \lim_{n \to \infty} k^{-n} = \infty
\]

\[
k < 1, \quad \lim_{n \to \infty} k^{-n} = 0
\]

\[
k = 1, \quad P_n = P_0 \quad \forall n
\]
Problems with this model are similar to those we encountered with the logistic differential equation.

One way to improve the model is to make some additional assumptions.

1. For small populations, the populations at consecutive time steps (consecutive generations) are proportional.

2. For large populations, all resources (space, food, etc.) are used up and the entire population will die out in the next time step.
To incorporate these assumptions we have:

\[ P_{n+1} = kP_n \left(1 - \frac{P_n}{M}\right) \]

\[ P_1 = kP_0 \]

\[ P_2 = kP_1 = k(kP_0) = k^2P_0 \]

\[ P_3 = kP_2 = k(k^2P_0) = k^3P_0 \]

\[ \vdots \]

\[ P_n = k^nP_0 \]

Where \( M \) is called the \textit{annihilation parameter}. If the population reaches \( M \) the species will die out.
In the model

\[ P_{n+1} = kP_n \left( 1 - \frac{P_n}{M} \right) \]

\( P_0 \) is the initial population and \( P_n \) is the population at the end of generation \( n \).

If \( P_n \) is small the difference equation becomes \( P_{n+1} \approx kP_n \). but if \( P_n > M \), then \( P_{n+1} \leq 0 \) which means the species is extinct.
To simplify our calculations we will assume that $P_n$ represents the percentage of the maximum population alive at generation $n$.

We assume $M = 1$ and $0 < P_n < 1$. Also $P_n \leq 0$ represents extinction and $P_n = 1$ is the maximum population.

The model becomes $P_{n+1} = kP_n \left(1 - P_n\right)$.
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Oscillating population

K=3.2
Cyclic population behavior

K = 3.5
K=3.9

No population pattern
**Iteration**

Finding successive population values is just iterating a quadratic function in the form

\[ L_k(x) = kx(1-x) \]
This is called the *Logistics Function*. To iterate this function we start with some initial population \( P_0 \) and ...

\[
P_1 = L_k \left( P_0 \right)
\]

\[
P_2 = L_k \left( P_1 \right)
\]

\[
P_3 = L_k \left( P_3 \right)
\]

\[\vdots\]

\[
P_n = L_k \left( P_{n-1} \right)
\]

The list of numbers \( P_0, P_1, P_2, \ldots \) is called the *orbit* of \( P_0 \) under the function \( L_k \). The initial value \( P_0 \) is called the *seed* for the orbit.
The basic goal in discrete dynamics is to predict the fate of orbits for a given function. That is, what happens to the orbits as $n \to \infty$?

Consider the example $F(x) = x^2$

Note that the seeds $x_0 = 0$ and $x_0 = 1$ are fixed points since $F(0) = 0$ and $F(1) = 1$. That is, their orbits are $0, 0, 0, 0, \ldots$ and $1, 1, 1, 1, \ldots$ respectively.
What about the orbit for $x_0 = -1$?
It looks like $-1, 1, 1, 1, \ldots$

Thus it becomes fixed since $F(-1) = 1$ is a fixed point.
For any other seed \( x_0 \) either the orbit tends to the fixed point at 0 or the orbit \( \rightarrow \infty \)

If \( x_0 = \frac{1}{2} \) the orbit is \( \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \ldots, \frac{1}{2^{2n}} \), which tends to zero as \( n \rightarrow \infty \). The fate for any seed \( x_0 \) with \( |x_0| < 1 \) is the same.
One the other hand, if $|x_0| > 1$ the orbit becomes arbitrarily large ...

if $x_0 = 2$ the orbit is $2, 4, 16, \ldots, 2^{2^n}, \ldots$

which $\to \infty$ as $n \to \infty$. 
Cycles.
In a discrete dynamical system, many different types of orbits are possible.

Consider \( G(x) = x^2 - 1 \). The orbit of \( x_0 = 0 \) lies on a cycle of period 2.

If we choose the seed \( x_0 = \sqrt{2} \), then this orbit is eventually periodic.
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This orbit never really reaches the cycle at 0 and at 1. The orbit comes arbitrarily close to these numbers but round-off eventually forces display of 0 and -1.

Thus the orbit is not periodic - it just tends to the cycle of period 2.

Though technically different, in practice the fate of these orbits is essentially the same.
Example problem.
Compute the orbit of $x_0 = 0$ for the difference equation

$$x_{n+1} = -x_n^2 + x_n + 2$$

Determine whether the orbit is fixed, cycles with some period, is eventually periodic, tends to infinity, or is none of these.
The sequence

\[
x_0 = 0, \\
x_1 = -x_0^2 + x_0 + 2 = 2, \\
x_2 = -x_1^2 + x_1 + 2 = 0, \\
x_3 = -x_2^2 + x_2 + 2 = 2
\]

shows that the orbit is periodic of period 2.
Finding fixed points.

Given a function $F$, to find the fixed points, we solve the equation $F(x) = x$.

Example — suppose $F(x) = x^2 - 6$. Then we must solve

$$x^2 - 6 = x \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0$$

and the fixed points for $F$ are $x = -2, 3$. 
Fixed points for the logistic equation.

We solve \( kx(1-x) = x \Rightarrow x = 0, \ x = \frac{k-1}{k} \).

For our population model we assume that \( 0 \leq x \leq 1 \) so the second fixed point is positive only if \( k > 1 \).

Also \( \frac{k-1}{k} < 1 \ \forall \ k > 1 \)
**Finding cycles.**

Notation: Let $F^n$ represent the $n$–th iterate of the function $F$.

Then $F^2(x) = F(F(x))$, $F^3(x) = F(F(F(x)))$, etc.
So, for some examples, if $F(x) = x^3$

then $F^2(x) = F(x^3) = (x^3)^3 = x^9$

and if $G(x) = (x-1)^2$

then $G^2(x) = G(G(x)) = G((x-1)^2) = ((x-1)^2 - 1)^2$

$= (x^2 - 2x + 1 - 1)^2 = (x^2 - 2x)^2 = x^4 - 4x^3 + 4x^2$
Example problem

Find all fixed points and periodic points of period 2 for the function $F(x) = x^2 - 3$. 
To find the fixed points we solve

\[ F(x) = x \text{ or } x^2 - 3 = x, \]

or

\[ x^2 - x - 3 = 0 \Rightarrow x = \frac{1 \pm \sqrt{13}}{2} \]

are the fixed points.
For periodic points of period 2 we first compute

\[ F^2(x) = (x^2 - 3)^2 - 3 = x^4 - 6x^2 + 6 \]

Then solve \( F^2(x) = x \) or \( x^4 - 6x^2 - x + 6 = 0 \)

We already know two solutions of this equation, they are the fixed points from \( x^2 - x - 3 = 0 \).
We're solving $x^4 - 6x^2 - x + 6 = 0$

We already know two solutions of this equation because any $x$-value satisfying $F(x) = x$ also satisfies $F^2(x) = x$.

Now by long division

$$x^4 - 6x^2 - x + 6 = (x^2 - x - 3)(x^2 + x - 2)$$

The roots of the quadratic factor $x^2 + x - 2$ give us the period 2 points $x = -2, 1$
Example problem

Describe the fate of the orbit of any seed under iteration of \( F(x) = x^3 \)
First to find the fixed points we solve

\[ F(x) = x \text{ or } x^3 = x. \]

So \( x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0 \Rightarrow x = 0, \pm 1 \) are the fixed points.
Then

\[ x_0 = F(x) = x, \]
\[ x_1 = F^2(x) = F(F(x)) = x^3, \]
\[ x_2 = F^3(x) = x^9, \]
\[ \ldots, \quad x_n = x^{3^n}. \]

Thus, given any real number \( x \), \( F^n(x) = x^{3^n} \)

Which for \( |x| < 1 \) tends to zero
and \( |x| > 1 \) tends to \( +\infty \) or \( -\infty \).
Example problem

Describe the fate of the orbit of the seed 2/7 under iteration of

\[ T(x) = \begin{cases} 
2x, & \text{if } x < \frac{1}{2} \\
2 - 2x, & \text{if } x \geq \frac{1}{2}
\end{cases} \]
The orbit is

$$\frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \frac{2}{7}, \ldots$$

so \(\frac{2}{7}\) is periodic with period 3 since

$$T\left(\frac{2}{7}\right) = \frac{4}{7},$$
$$T\left(\frac{4}{7}\right) = \frac{6}{7},$$
$$T\left(\frac{6}{7}\right) = \frac{2}{7}, \ldots$$
Example problem, Fibonacci numbers

Sometimes we can actually solve a difference equation explicitly.

Consider the initial-value difference equation problem:

\[ F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, F_1 = 1 \]
Assume the equation has a solution of the form $F_n = \varphi^n$, where $\varphi$ is a constant to be determined. Then $F_{n+2} = \varphi^{n+2}$, $F_{n+1} = \varphi^{n+1}$. Plugging these into our difference equation gives

$$\varphi^{n+2} = \varphi^{n+1} + \varphi^n \Rightarrow \varphi^{n+2} - \varphi^{n+1} - \varphi^n = 0$$

Factoring gives

$$\varphi^n (\varphi^2 - \varphi - 1) = 0 \Rightarrow \varphi^2 - \varphi - 1 = 0$$

with solutions

$$\varphi = \frac{1 \pm \sqrt{5}}{2} \Rightarrow F_n = \varphi^n = \left(\frac{1 \pm \sqrt{5}}{2}\right)^n$$
This is a second order difference equation so we expect a general solution of the form

\[ F_n = k_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + k_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

Applying the initial conditions

\[ F_0 = 0 = k_1 + k_2 \Rightarrow k_2 = -k_1 \]
\[ F_1 = 1 = k_1 \left( \frac{1 + \sqrt{5}}{2} \right) + k_2 \left( \frac{1 - \sqrt{5}}{2} \right) = k_1 \left( \frac{1 + \sqrt{5}}{2} \right) - k_1 \left( \frac{1 - \sqrt{5}}{2} \right) \]
\[ = k_1 \left[ \left( \frac{1 + \sqrt{5}}{2} \right) - \left( \frac{1 - \sqrt{5}}{2} \right) \right] = k_1 \sqrt{5} \Rightarrow k_1 = \frac{1}{\sqrt{5}}, \quad k_2 = -\frac{1}{\sqrt{5}} \]

So our solution is

\[ F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n \quad n = 0, 1, 2, \ldots \]
This is a second order difference equation so we expect a general solution of the form

\[ F_n = k_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + k_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n \]
End Presentation