Where are Differential Equations used?

1. Mathematical modeling
2. Engineering
3. Physics
4. Life sciences
5. Chemistry
6. Business
Differential equations form part of a larger subject known as *Dynamics*. This is the subject that deals with change, with systems that evolve in time.

Ordinary differential equations describe the evolution of systems in *continuous* time.
Differential equations come in two flavors, ordinary differential equations and partial differential equations.

An example of an ordinary differential equation is the so-called damped harmonic oscillator

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0 \quad m, k, b \in \mathbb{R}, m \neq 0$$

There is only one independent variable, $t$
An example of a partial differential equation is the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

This equation has both time $t$, and space $x$ as independent variables.

We will deal with ordinary differential equations exclusively in this course.
Mathematical Modeling and Differential Equations ...

- What is a mathematical model?
- How are differential equations used to build mathematical models?
- What good is all of this stuff anyway?
Mathematical Models are used to

- Predict the future!
- Study time varying phenomena
  - Weather prediction
  - Analysis of structures
  - Medical applications
  - Military applications
  - Video games
  - Population studies
    - Predator-prey systems
Building a Mathematical Model I

- Clearly state all assumptions being made
- Completely describe all variables and parameters to be used in the model
- Use assumptions to derive equations relating variables and parameters
- Analyze/solve the resulting equations
- Compare results with observed data
Building a model II

Do the results agree with real data?

If so, confidence in model grows.

If not, improve assumptions, variables, parameters, and/or equations …

Then repeat the previous steps.
Building a model III

- Models not designed to be “exact” copies of the real thing.
- Often simple models are better and easier to use than more precise models.
Our approach will include ...

- Analytical methods
- Numerical methods
- Qualitative methods
Analytical Methods

We look for closed form solutions.
A typical first order initial value problem:
We want to find a function, \( y(t) \), that satisfies the differential equation and the initial condition.

\[
\frac{dy}{dt} = f(y, t), \quad y(t_0) = y_0
\]
Very few differential equations have closed form solutions

Even if they do, often not very informative
Numerical Methods ...

- Large number of methods available
- We’ll use Euler’s method

Advantage
  Simple to understand/apply

Disadvantage
  Only an approximation to the actual solution
Qualitative Methods ...

- Classification of equilibrium points
- Vector/Direction fields
- Phase lines
- Phase planes
- Eigenanalysis
- Stability
- Linearization
We now state a theorem giving sufficient conditions for the existence and uniqueness of a solution for the first-order equation.

**Existence and Uniqueness Theorem**

Let the equation \( \frac{dx}{dt} = f(t,x) \) be given, where \( f(t,x) \) is defined in some domain \( B \). Suppose in addition that \( f(t,x) \) and \( \frac{\partial f}{\partial x} \) are defined and continuous in \( B \).

Then for every point \( (t_0, x_0) \) in \( B \) there exists a unique solution \( x = \varphi(t) \) of \( \frac{dx}{dt} = f(t,x) \) satisfying \( \varphi(t_0) = x_0 \) and defined in some neighborhood of \( (t_0, x_0) \).
Some remarks are in order. First of all, by *unique* is meant the following.

If two solutions $x = \varphi(t)$ and $x = \alpha(t)$ of the equation

$$\frac{dx}{dt} = f(t, x)$$

both satisfy $\varphi(t_0) = \alpha(t_0) = x_0$, then these solutions are identical in their common interval of definition.

So the theorem states that through every point of $B$ there passes *one and only one* integral curve/trajectory.
The E-U theorem is useful in the following sense:
Suppose by some technique we are able to find a family $K$ of solutions of $\frac{dx}{dt} = f(t, x)$. Furthermore, given a point $(t_0, x_0)$ in $B$, there is an element $\varphi$ in $K$ satisfying $\varphi(t_0) = x_0$.

If the hypotheses of the E-U theorem are satisfied, then by uniqueness $K$ must describe all solutions, and we need look no further for other solutions.
\[ \frac{dy}{dt} = y - t \]

- \( \min y = -3 \)
- \( \max y = 3 \)
- \( \min t = -5 \)
- \( \max t = 5 \)
- \( y_0 = 1.16 \)
- \( t_0 = 0.42 \)
- \( \delta t = 0.05 \)
The class of differential equations for which an explicit solution can be found is small.

If we have a differential equation for which we know a solution exists, we can proceed to investigate properties of the solution, regardless of whether we know its explicit form.

The qualitative study of solutions of differential equations thus depends on existence.
Uniqueness would be of importance if we wished to approximate the solution numerically.

If two solutions passed through a point, then successive approximations could very well jump from one solution to the other – with unpredictable consequences.
End Presentation