A simple separable example ...

\[ \frac{dP}{dt} = kP, \quad P(t_0) = P_0, \quad k > 0 \]

Which has the closed form solution ...

\[ P(t) = P_0 e^{k(t-t_0)} \]
How do we solve \( \frac{dP}{dt} = kP, \quad P(t_0) = P_0, \quad k > 0? \)

This is called an *initial-value problem*.
To solve the equation, "separate the variables" to get

\[
\frac{dP}{P} = k\,dt
\]

and integrate both sides:

\[
\int \frac{dP}{P} = \int k\,dt
\]
Integration gives: \( \ln |P(t)| = kt + c, \)
where \( c \) is a constant of integration. We can discard absolute value since we are assuming \( P(t) \) represents a population at time \( t \) : \( P(t) > 0, \quad \forall t. \)

Exponentiating both sides,

\[
P(t) = e^{kt+c} = e^c e^{kt} = Me^{kt}, \quad \text{where } e^c = M
\]
The function \( P(t) = Me^{kt} \)

is called the \textit{general solution} for the differential equation.

Since we are given an initial condition, \( P(t_0) = P_0 \), we can find a \textit{particular solution} for the initial value problem.
Now apply the initial condition, \( P(t_0) = P_0 \), to determine the constant of integration \( M \):

\[
P(t_0) = P_0 = Me^{kt_0} \Rightarrow M = P_0 e^{-kt_0}
\]

Finally, the particular solution is

\[
P(t) = P_0 e^{kt-kt_0} = P_0 e^{k(t-t_0)}
\]
Suppose we have

\[
\frac{dP}{dt} = kP, \quad P(t_0) = P_0
\]

with \( t_0 = 0, \ P_0 = 50, \ k = 1.105 \)

With closed form solution

\[
P(t) = 50e^{1.105t}
\]
Sample solution curves for \( \frac{dP}{dt} = kP, \quad P(t_0) = P_0 \)

with \( t_0 \sim -5.00, \quad P_0 = \text{various values}, \quad k = 1.105 \)

\[
P(t) = P_0(t)e^{1.105t}
\]
\[ P(0) = 1.5 \]
Population models

Exponential model (unlimited growth)

Main assumption: human population change is proportional to the current population.

\[
\frac{dP}{dt} = kP, \quad P(t_0) = P_0, \quad k > 0
\]

and \( P(t) = P_0 e^{kt-t_0} \)
Cane Toad - *Bufo marinus*
Type: Amphibian
Diet: Omnivore

Average life span in the wild: 5 to 10 years

Size: 4 to 6 in.
Weight: 2.9 lbs.

Group name: Knot or nest.
Ok, back to humans. How good is this model?

To evaluate it, we need to compare our predictions with actual data.

\[ t_0 = 1965, \quad P_0 = 3.34 \times 10^9, \quad k = 0.02 \]

\[ P(t) = (3.34 \times 10^9) e^{0.02(t-1965)} \]

One way to evaluate the model is to compute the population doubling time.

Observed doubling time is ~35 years.
Doubling time calculations

\[ 2P_0 = P_0 e^{0.02T} \implies e^{0.02T} = 2 \implies T = \frac{\ln 2}{0.02} \implies T \approx 34.6 \]

This is in excellent agreement with the observed value of 35 years.

However, what about the distant future?
## Exponential model prediction results

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>2515</td>
<td>200,000 billion</td>
</tr>
<tr>
<td>2625</td>
<td>1,800,000 billion</td>
</tr>
<tr>
<td>2660</td>
<td>3,600,000 billion</td>
</tr>
</tbody>
</table>
## Exponential model results - space

<table>
<thead>
<tr>
<th>Year</th>
<th>Area per person</th>
</tr>
</thead>
<tbody>
<tr>
<td>2515</td>
<td>9.3 square feet</td>
</tr>
<tr>
<td>2625</td>
<td>1 square foot</td>
</tr>
<tr>
<td>2660</td>
<td>Two deep on each others shoulders !!</td>
</tr>
</tbody>
</table>
Locally the model is reasonable, globally the model is unrealistic.

The model requires modification to produce more reasonable long-term predictions.
For large populations these linear models cannot be accurate.

They do not reflect the fact that individual members of the population are now competing with each other for the limited living space, natural resources, and available food.
Thus we add a competition term to our linear differential equation.

A reasonable choice of a competition term is

\[-bP^2\]

where \( b \) is a positive constant, since the statistical average of the number of encounters of two members of the population per unit time is proportional to \( P^2 \).
Improved model

\[ \frac{dP}{dt} = aP - bP^2, \quad P(t_0) = P_0, \ a, b > 0 \]

\( a \) and \( b \) are called the vital coefficients of the population. This equation is often written

\[ \frac{dP}{dt} = aP \left( 1 - \frac{P}{N} \right), \quad \text{where} \ N = \frac{a}{b}. \]
To solve the initial-value problem analytically

\[ \frac{dP}{dt} = aP - bP^2, \quad P(t_0) = P_0, \]

separate variables and integrate as before to get

\[ \int \frac{dP}{aP - bP^2} = \int dt \]

We handle the left hand side by partial fraction decomposition and we get the solution ...
The closed form solution is

\[ P(t) = \frac{aP_0}{bP_0 + \left(a - bP_0\right)e^{-a(t-t_0)}} \]

What can we learn from this solution?
At least the population is finite!

\[
P(t) = \frac{aP_0}{bP_0 + (a-bP_0)e^{-a(t-t_0)}}
\]

\[
\Rightarrow \lim_{t \to \infty} P(t) = \frac{a}{b}
\]

regardless of value of \( P(t_0) = P_0 \)
Ecologists estimate that the natural values for $a$ and $b$ are:

$$a = 0.029 \text{ and } b = 2.695 \times 10^{-12}$$

So

$$N = \frac{a}{b} = \frac{0.029}{2.695 \times 10^{-12}} = 10.76 \text{ Billion}$$

Thus the *carrying capacity* of the earth is

$\sim 10.76$ billion people. How good is this estimate?
Population graph - our model

- $2 \times 10^9$
- $4 \times 10^9$
- $6 \times 10^9$
- $8 \times 10^9$
- $1 \times 10^{10}$
World Population Growth Rate: 1950-2050

Source: U.S. Census Bureau, International Data Base, April 2005 version.
Annual World Population Change: 1950-2050

Source: U.S. Census Bureau, International Data Base, April 2005 version.
World Population: 1950-2050

Source: U.S. Census Bureau, International Data Base, April 2005 version.
Our estimate - same time interval
Growth Rate

Population in the world is currently growing at a rate of around 1.14\% per year.

The average population change is currently estimated at around 80 million per year.

(Data from the Census Bureau website)
Annual growth rate reached its peak in the late 1960s, when it was at 2% and above.

The rate of increase has therefore almost halved since its peak of 2.19 percent, which was reached in 1963.
The annual growth rate is currently declining and is projected to continue to decline in the coming years.

Currently, it is estimated that it will become less than 1% by 2020 and less than 0.5% by 2050.
This means that the world population will continue to grow in the 21st century, but at a slower rate compared to the recent past.

World population has doubled in 40 years from 1959 (3 billion) to 1999 (6 billion). (That’s a 100% increase)
It is now estimated that it will take a further 43 years to increase by another 50%, to become 9 billion by 2042.

The latest United Nations projections indicate that world population will nearly stabilize at just above 10 billion persons after 2062.
Current population estimates

Births this year: 56,739,197
Deaths this year: 23,411,431
Population growth this year: 33,328,204
As of: May 29, 2014
Current population estimates

U.S.: 322,359,556
World: 7,236,651,034
as of: May 29, 2014