Extending the population model to include harvesting
Consider the logistic growth model with a harvesting term:

\[
\frac{dP}{dt} = kP(M - P) - a
\]

where again the variables are \( P(t) \) for population and \( t \) for time. Here the parameters are:

- \( M \), the theoretical equilibrium population,
- \( k \) the growth constant,
- \( a \) is the (constant) harvesting rate.
\[ \frac{dP}{dt} = kP(M - P) - a \]

The equation is autonomous and nonlinear.

We can find a closed form solution by separation of variables and partial fractions. Here we will look at the geometry instead.
If \( a = 0 \), the equilibria are \( P = 0 \) and \( P = M \).

For small \( a \) the equilibria are found by solving the quadratic equation \( aP(M - P) - a = 0 \), whose solutions are

\[
P = \frac{M}{2} \pm \frac{1}{\sqrt{k}} \sqrt{\frac{kM^2}{4} - a}.
\]

For \( a > \frac{kM^2}{4} \), there are no equilibria.

Between the equilibria \( \frac{dP}{dt} > 0 \) and \( \frac{dP}{dt} \) is negative elsewhere.
As an application, suppose there is a lake which is stocked with fish.

Through research biologists have determined that the logistic equation applies to the fish population living and reproducing in the lake:

\[ \frac{dP}{dt} = kP(M - P), \]

where to simplify the calculations we set parameter \( M = 1 \) (number of fish in 1000s) and parameter \( k \) to equal 1 unit to get \[ \frac{dP}{dt} = P(1 - P). \] Time \( t \) is measured in months.
We want to determine what impact various fishing (harvesting) rates, \( a \), in fish/month will have on the population.

Obviously we want to be sure we have a sustainable population long term.

Our differential equation becomes

\[
\frac{dP}{dt} = P(1 - P) - a
\]
\[
\frac{dP}{dt} = P(1 - P) - a
\]

If \( a = 0 \), the equilibria are \( P = 0 \) and \( P = 1 \). For small \( a \) the equilibria are found by solving the quadratic equation \( P(1 - P) - a = 0 \), whose solutions are

\[
P = \frac{1}{2} \pm \sqrt{\frac{1}{4} - a}
\]

For \( a > \frac{1}{4} \), there are no equilibria.

Between the two equilibria \( \frac{dP}{dt} > 0 \) and \( \frac{dP}{dt} \) is negative elsewhere.
No harvesting
No harvesting: solutions
Harvesting rate = 0.12
Harvesting rate = 0.12: solutions
Harvesting rate = 0.21
Harvesting rate = 0.21: solutions
Harvesting rate = 0.25
Harvesting rate = 0.25: solutions
Harvesting rate = 0.35
Harvesting rate = 0.35: solutions
So, what happened?

We started with a logistic population model with harvesting:

\[
\frac{dP}{dt} = kP(M - P) - a
\]

When there was no harvesting we had an equilibrium solution at \( P = M \).

This represents a constant solution so that as \( t \to \infty \), \( P(t) \to M \).
An equilibrium solution is also called a *fixed point* for the differential equation.

Thus a fixed point is a value of the dependent variable $P$ which makes the slope zero *for all values* of the independent variable $t$.

A fixed point is called _stable_ if any solution with value close enough to $P$ has subsequent values approaching $P$. Such a stable point is also called a _sink_ or an _attractor_. 
Our differential equation also contained parameters $k, M,$ and $a.$

These are quantities in the differential equation that are constant. However, when we label a constant a "parameter" we're usually interested in what happens when we change its value!!
When we allowed the harvesting parameter to move through a sequence of values we observed that the solutions for the differential equation underwent change.

In fact, once the harvesting parameter reached a critical value ($a = 0.25$) a bifurcation took place.

A bifurcation occurs at a parameter value $c$ if the qualitative behavior of the differential equation changes as the parameter increases through $c$. 
For our model the bifurcation point is $a = 0.25$ as that is the value for which there is a large qualitative change in the solutions for the differential equation.

That is, at $a = 0.25$, the two equilibria collapsed into a single equilibrium point and then as $a$ increased beyond 0.25, the equilibria vanished!
End Presentation